Finite-time extremum seeking control for a class of unknown static maps

Guay, Martin; Benosman, Mouhacine

TR2020-117 August 28, 2020

Abstract

This paper proposes an extremum-seeking control design that achieves finite-time stability of the optimum of an unknown measured cost function. The finite-time extremum-seeking control technique is shown to achieve finite-time practical stability of the optimum of the cost function. The main characteristic of the proposed extremum seeking control approach is that the corresponding target averaged system achieves finite-time stability. A simulation study is presented to demonstrate the effectiveness of the approach.

International journal of adaptive control and signal processing

Received: Added at production | Revised: Added at production | Accepted: Added at production
DOI: xxx/xxxx

ARTICLE TYPE

Finite-time extremum seeking control for a class of unknown static maps

Martin Guay*[1] | Mouhacine Benosman[2]

[1]Department of Chemical Engineering, Queen's University, Kingston, ON,Canada
[2]Mitsubishi Electric Research Laboratories (MERL), Cambridge, MA,USA

Summary

This paper proposes an extremum-seeking control design that achieves finite-time stability of the optimum of an unknown measured cost function. The finite-time extremum-seeking control technique is shown to achieve finite-time practical stability of the optimum of the cost function. The main characteristic of the proposed extremum seeking control approach is that the corresponding target averaged system achieves finite-time stability. A simulation study is presented to demonstrate the effectiveness of the approach.

KEYWORDS:
Extremum-seeking control, Finite-time stability, Nonlinear systems

1 | INTRODUCTION

Extremum seeking control (ESC) is a feedback control mechanism designed to drive an unknown nonlinear dynamical system to the optimum of a measured variable of interest[1]. The basic stability properties of ESC were first outlined in[2] and[3]. Based on this initial theoretical work, a vast and growing literature has been evolving to complement, generalize and improve the basic schemes.

Finite-time stability in control systems is a desirable property in many applications where the timing of control tasks is critical. One typical example is the control of batch systems which are designed to operate over finite-time horizons[4 5]. Other typical problems, amongst others, include finite-time distributed control of multi-agent systems[6 7], finite-time synchronization[8] and rendezvous problems[9]. Several studies have been recently conducted in the design and analysis of finite-time control systems. The general stability conditions were first developed in[10]. The finite-time stabilization of a class of controllable systems was considered in[11]. The output feedback finite-time stabilization of nonlinear systems was considered in[12] for the local case. A perspective of global finite-time stabilization was provided in[13]. Robust finite-time stabilization was treated in[14]. The concept of finite-time input to state stability (FTISS) was presented in[15]. This study provides a complete characterization of finite-time nonlinear systems subject to external inputs. The finite-stability property is usually associated with dynamical control systems that are either non-Lipschitz or discontinuous. In most existing work, finite-time stable systems are closely related to classes of nonlinear systems with continuous right hand side[10]. In a similar fashion, it can be shown that finite-time stabilization can be achieved using continuous feedback controllers[11].

The problem of finite-time optimization using gradient-based descent algorithms was addressed in[18]. A comprehensive stability analysis was provided to address the non-Lipschitz nature of finite-time gradient systems. In[19,20], a class of discontinuous dynamical systems was proposed for the design of continuous-time optimization algorithms with finite-time convergence and prescribed convergence time. In contrast to the gradient based algorithms, this class of optimization algorithms incorporates second order information of the cost function. Using this information, improved performance is achieved for systems with time-varying cost functions.

In this manuscript, we propose an ESC design technique that can achieve finite-time stability in the practical sense. Given a measured cost function with an unknown mathematical formulation, the objective of this study is to design an ESC that brings the system to a neighborhood of the unknown optimum value of the input in finite-time. The finite-time can be prescribed by tuning the parameters of the finite-time ESC system.

In the analysis of the proposed ESC, it is shown that the proposed technique yields an ESC system that is continuous everywhere. In addition, it is shown that the resulting averaged system has a finite-time stable equilibrium at the unknown optimum of the measured cost function. The detailed analysis of the resulting averaged nonlinear system is based on the concept of finite-time input to state stability (FTISS) first introduced by Hong *et al*[15]. In particular, the small-gain approach proposed by Hong *et al*[15] is applied to demonstrate the finite-time stability of the averaged ESC. A classical averaging analysis result[22] is applied to show that the closed-loop finite-time ESC system achieves finite-time practical stability of the optimum. The paper is structured as follows. The problem formulation is given in Section 2. In Section 3, a target averaged finite-time ESC system is proposed. The proposed ESC is presented in Section 3.3. A simulation study is given in Section 4. Brief conclusions and a discussion of future research work is presented in Section 5.

2 | PROBLEM FORMULATION

In this study, we consider a class of unknown nonlinear systems described by the following dynamical system:

$$\dot{x} = u \tag{1a}$$

$$y = h(x) \tag{1b}$$

where $x \in \mathbb{R}$ are the state variables, $u \in \mathbb{R}$ is the input variable, and $y \in \mathbb{R}$ is the output variable. It is assumed that the function $h : \mathbb{R} \to \mathbb{R}$ is sufficiently smooth. The function h, is assumed to be unknown. It has an unknown minimizer x^* with an optimal value $y^* = h(x^*)$.

We make the following assumptions concerning the measured cost function, $h(x)$.

Assumption 1. The function $h(x)$ is such that its gradient vanishes only at the minimizer x^*, that is:

$$\left.\frac{\partial h}{\partial x}\right|_{x=x^*} = 0.$$

The Hessian at the minimizer is assumed to be positive and nonzero. In particular, there exists a positive constant α_h such that

$$\frac{\partial^2 h(x)}{\partial x \partial x^T} \geq \alpha_h I$$

for all $x \in \mathcal{X} \subset \mathbb{R}$.

The objective of this study is to develop an ESC design technique that guarantees finite-time convergence to the unknown minimizer, x^*, of the measured function $y = h(x)$.

3 | FINITE TIME EXTREMUM SEEKING CONTROLLER DESIGN AND ANALYSIS

3.1 | Finite-time Stability

We first provide a formal definition of finite-time stability (as stated in[15]) that will be used throughout the manuscript.

Consider the system:

$$\dot{X} = F(X) \tag{2}$$

where $X \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to X.

The continuity of the right hand side of (2) guarantees existence of at least one solution, which is possibly non-unique. We denote by $\mathcal{X}(t, t_0, X_0)$ the set of all solutions with initial conditions $X(t_0) = X_0$ for $t \geq t_0$. The set of all solutions of system (2) at time t is denoted by $X(t)$. It is assumed that the equilibrium $X_0 = 0$ is a unique solution of the system in forward time.

Definition 1. The equilibrium $X = 0$ of (2) is said to be finite-time locally stable if it is Lyapunov stable and such that there exists a settling-time function

$$T(X_0) = \inf \left\{ \bar{T} \geq t_0 \,\middle|\, \lim_{t \to \bar{T}} X(t) = 0\,;\; X(t) \equiv 0,\, \forall t \geq \bar{T} \right\}$$

in a neighbourhood U of $X = 0$. It is globally finite-time stable if $U = \mathbb{R}^n$.

A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a called a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. It is a class $\mathcal{K}_\infty$ function if it is class $\mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$.

A continuous function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a generalized class $\mathcal{K}$ function if $\phi(0) = 0$ and

$$\begin{cases} \phi(s_1) > \phi(s_2) & \text{if } \phi(s_1) > 0,\; s_1 > s_2 \\ \phi(s_1) = \phi(s_2) & \text{if } \phi(s_1) = 0,\; s_1 > s_2. \end{cases} \tag{3}$$

A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a generalized $\mathcal{KL}$ function if, for each fixed $t \geq 0$, the function $\beta(s,t)$ is a generalized $\mathcal{K}$ function and each fixed $s \geq 0$, the function $\beta(s,t)$ is such that $\lim_{t\to T} \beta(s,t) = 0$ for $T \leq \infty$.

Definition 2. System (2) is finite-time stable if there exists a generalized $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that every solution $X(t)$ satisfies:

$$\|X(t)\| \leq \beta(\|X(0)\|, t) \tag{4}$$

with $\beta(r,t) \equiv 0$ when $t \geq \bar{T}(r)$ with $\bar{T}(r)$ continuous with respect to r and $\bar{T}(0) = 0$.

Definition 3. Let $V(X)$ be a continuous function. It is called a finite-time Lyapunov function if there exists class $\mathcal{K}_\infty$ functions ϕ_1 and ϕ_2 and a class $\mathcal{K}$ function ϕ_3 such that:

$$\phi_1(\|X\|) \leq V(X) \leq \phi_2(\|X\|)$$

and

$$D^+V(X(t)) = \limsup_{s\to 0^+} \frac{V(X(t+s)) - V(X(t))}{s} \leq -\phi_3(\|X\|)$$

where, in addition, ϕ_3 satisfies:

$$c_1 V(X)^a \leq \phi_3(\|X\|) \leq c_2 V(X)^a$$

for some positive constants $a < 1$, $c_1 > 0$ and $c_2 > 0$.

Next, we consider the system:

$$\dot{X} = F(X, \upsilon(t)) \tag{5}$$

where $X \in \mathbb{R}^n$. The function $\upsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is measurable and locally essentially bounded and the vector value function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous in X and $\upsilon(t)$.

Definition 4. System (5) is finite-time input-to-state stable (FTISS) if there exists a generalized $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}$ function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that every solution $X(t)$ satisfies:

$$\|X(t)\| \leq \beta(\|X(0)\|, t) + \alpha(\|\upsilon(t)\|_\infty) \tag{6}$$

with $\beta(r,t) \equiv 0$ when $t \geq \bar{T}(r)$ with $\bar{T}(r)$ continuous with respect to r and $\bar{T}(0) = 0$.

Definition 5. Let $V(X)$ be a continuous function. It is called an FTISS Lyapunov function if there exists class $\mathcal{K}_\infty$ functions ϕ_1 and ϕ_2 and class $\mathcal{K}$ functions ϕ_3 and ϕ_4 such that:

$$\phi_1(\|X\|) \leq V(X) \leq \phi_2(\|X\|)$$

and,

$$\|X(t)\| \geq \phi_4(\|\upsilon(t)\|) \Rightarrow D^+V(X(t)) \leq -\phi_3(\|X\|)$$

where, in addition, ϕ_3 satisfies:

$$c_1 V(X)^a \leq \phi_3(\|X\|) \leq c_2 V(X)^a$$

for some positive constants $a < 1$, $c_1 > 0$ and $c_2 > 0$.

We define semi-global practical finite-time stability as follows.

Definition 6. System (2) is semi-globally practically finite-time stable if there exists a generalized $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a positive constant $\zeta > 0$ such that every solution $X(t)$ starting in $\mathcal{X}$ satisfies:

$$\|X(t)\| \leq \beta(\|X(0)\|, t) + \zeta \tag{7}$$

with $\beta(r, t) \equiv 0$ when $t \geq \bar{T}(r)$ with $\bar{T}(r)$ continuous with respect to r and $\bar{T}(0) = 0$.

The following two theorems are proven in Hong *et al*[15].

Theorem 1. [15] System (5) is FTISS if it admits an FTISS Lyapunov function.

Finally, we consider the interconnection of two FTISS systems:

$$\begin{aligned} \dot{X}_1 &= F_1(X_1, X_2) \\ \dot{X}_2 &= F_2(X_2, X_1) \end{aligned} \tag{8}$$

where $X_1 \in \mathbb{R}^{n_1}$ and $X_2 \in \mathbb{R}^{n_2}$ with $F_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ and $F_2 : \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ are continuous in X_1 and X_2 with a unique equilibrium at $X_1 = 0$ and $X_2 = 0$. As above, it is assumed that the equilibrium is the unique solution to (8) forward in time. In the following, we let $\alpha \circ \beta$ denote the composition of two functions α and β.

Theorem 2. [15] Suppose (8) are FTISS, with X_2 as an input for the X_1 subsystem and with X_1, the input of the X_2 subsystem. Suppose that the solutions of each system satisfy:

$$\begin{aligned} \|X_1(t)\| &\leq \beta_1(\|X_1(0)\|, t) + \alpha_1(\|X_2(t)\|_\infty), \\ \|X_2(t)\| &\leq \beta_2(\|X_2(0)\|, t) + \alpha_2(\|X_1(t)\|_\infty) \end{aligned}$$

where β_1 and β_2 are generalized $\mathcal{KL}$ functions and α_1 and α_2 are class $\mathcal{K}$ functions. If there exists class $\mathcal{K}_\infty$ functions ρ_1 and ρ_2 that satisfy:

$$(Id + \rho_2) \circ \alpha_2 \circ (Id + \rho_1) \circ \alpha_1(s) \leq s, \quad s \geq 0$$

then $X_1 = 0$, $X_2 = 0$ is a finite-time stable equilibrium of system (8).

3.2 | Proposed target average system

In the design of ESC, one seeks an averaged system that can be achieved using a judicious choice of dither signals. We consider the following system:

$$\begin{aligned} \dot{x} &= -\gamma(\xi)\xi \\ \dot{\xi} &= -K\gamma(\xi - \nabla h(x))(\xi - \nabla h(x)) \end{aligned} \tag{9}$$

where K is a controller gain to be assigned. As in[16,17], the function $\gamma(\xi)$ is given by:

$$\gamma(z) = \frac{c_1}{\|z\|^{\alpha_1}} + \frac{c_2}{\|z\|^{\alpha_2}}$$

where σ is a small positive constant, $\alpha_1 = \dfrac{q_1 - 2}{q_1 - 1}$ and $\alpha_2 = \dfrac{q_2 - 2}{q_2 - 1}$ for $q_1 \in (2, \infty)$ and $q_2 \in (1, 2)$.

The function $X_1(x, \xi) = \gamma(\xi)\xi$ is not locally Lipschitz continuous at $\xi = 0$ but it is continuous everywhere. Similarly, the function $X_2(x, \xi) = \gamma(\xi - \nabla h(x))(\xi - \nabla h(x))$ is also not locally Lipschitz continuous for $\forall x$ and $\forall \xi$ such that $\xi = \nabla h(x)$ but it is continuous at this point. It is locally Lipschitz everywhere else.

We define the state-space transformation $z = \xi - \nabla h(x)$ and rewrite the dynamics as:

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= -K\gamma(z)z + \nabla^2 h(x)\gamma(z + \nabla h(x)))(z + \nabla h(x))) \\ \frac{\mathrm{d}\nabla h}{\mathrm{d}t} &= -\gamma(z + \nabla h(x))\nabla^2 h(x)(z + \nabla h(x))). \end{aligned} \tag{10}$$

This system is such that it has a unique equilibrium at the point $z = 0$ and $\nabla h(x) = 0 \Rightarrow x = x^*$. It is also continuous everywhere and locally Lipschitz continuous away from the equilibrium.

Next we proceed to the stability analysis of system (9).

Theorem 3. Consider the nonlinear system (9). Let Assumption 1 be satisfied. Then the optimum x^* is a finite-time stable equilibrium of the system.

Proof: We first consider the function $V_1 = \frac{1}{2}z^2$. Its derivative along the trajectories of the system yields:

$$\dot{V}_1 = -K\gamma(z)z^2 + \gamma(z+\nabla h(x))\nabla^2 h(x)(z+\nabla h(x))z.$$

The function $\phi(z, \nabla h(x)) = \gamma(z+\nabla h(x))(z+\nabla h(x))$ is continuous. It is such that:

$$|\phi(z,\nabla h(x))| \leq \frac{c_1|z+\nabla h(x)|}{|z+\nabla h(x)|^{\alpha_1}} + c_2\frac{|z+\nabla h(x)|}{|z+\nabla h(x)|^{\alpha_2}} = c_1|z+\nabla h(x)|^{1-\alpha_1} + c_2|z+\nabla h(x)|^{1-\alpha_2}. \tag{11}$$

By the triangle inequality, one obtains:

$$|\phi(z,\nabla h(x))| \leq c_1|z|^{1-\alpha_1} + c_1|\nabla h(x)|^{1-\alpha_1} + c_2|z|^{1-\alpha_2} + c_2|\nabla h(x)|^{1-\alpha_2} \tag{12}$$

or,

$$|\phi(z,\nabla h(x))| \leq c_1\frac{|z|}{|z|^{\alpha_1}} + c_1\frac{|\nabla h(x)|}{|\nabla h(x)|^{\alpha_1}} + c_2\frac{|z|}{|z|^{\alpha_2}} + c_2\frac{|\nabla h(x)|}{|\nabla h(x)|^{\alpha_2}}. \tag{13}$$

By definition, the last inequality can be written as:

$$|\phi(z,\nabla h(x))| \leq \gamma(z)|z| + \gamma(\nabla h(x))|\nabla h(x)|.$$

Upon substitution, it follows that $\dot{V}_1$ fulfills the following inequality:

$$\begin{aligned}\dot{V}_1 &\leq -(K - \nabla^2 h(x))\gamma(z)z^2 + \gamma(\nabla h(x))|\nabla h(x)||z| \\ &\leq -(K-\nabla^2 h(x))(c_1|z|^{2-\alpha_1} + c_2|z|^{2-\alpha_2}) + (c_1|\nabla h(x)|^{1-\alpha_1} + c_2|\nabla h(x)|^{1-\alpha_2})|z|\end{aligned}$$

Next, we choose K such that $(K - \nabla^2 h(x)) \geq \alpha$:

$$\dot{V}_1 \leq -\alpha(c_1|z|^{2-\alpha_1} + c_2|z|^{2-\alpha_2}) + (c_1|\nabla h(x)|^{1-\alpha_1} + c_2|\nabla h(x)|^{1-\alpha_2})|z|$$

Let $c \in (0,1)$, we can rearrange the last inequality as:

$$\dot{V}_1 \leq -(1-c)\alpha(c_1|z|^{2-\alpha_1} + c_2|z|^{2-\alpha_2}) - \alpha c|z|\left(c_1|z|^{1-\alpha_1} + c_2|z|^{1-\alpha_2} - \frac{1}{c\alpha}(c_1|\nabla h(x)|^{1-\alpha_1} + c_2|\nabla h(x)|^{1-\alpha_2})\right)$$

Therefore, one obtains:

$$\dot{V}_1 \leq -(1-c)\alpha(c_1|z|^{2-\alpha_1} + c_2|z|^{2-\alpha_2}), \text{ if} |z| \geq \frac{1}{c\alpha}|\nabla h(x)|.$$

Using the definition of V_1, we finally get:

$$\dot{V}_1 \leq -2(1-c)\alpha\left(c_1 V_1^{1-\frac{\alpha_1}{2}} + c_2 V_1^{1-\frac{\alpha_2}{2}}\right) \equiv \phi_3(\|z\|), \text{ if} |z| \geq \frac{1}{c\alpha}|\nabla h(x)|.$$

Following Definition 5, it follows that V_1 is an FTISS Lyapunov function and therefore the z dynamics are $FTISS$ with input $\nabla h(x)$.

For the gradient dynamics $\nabla h(x)$, we consider the Lyapunov function $V_2 = \frac{1}{2}(\nabla h(x))^2$. Its rate of change is given by:

$$\dot{V}_2 = -\nabla^2 h(x)\gamma(z+\nabla h(x))(z+\nabla h(x))\nabla h(x).$$

As above, we write:

$$\dot{V}_2 = -\nabla^2 h(x)\left(c_1\frac{z+\nabla h(x)}{|z+\nabla h(x)|^{\alpha_1}} + c_2\frac{z+\nabla h(x)}{|z+\nabla h(x)|^{\alpha_2}}\right)\nabla h(x).$$

or,

$$\dot{V}_2 = -\nabla^2 h(x)\left(c_1\frac{z\nabla h(x) + \nabla h(x)^2}{|z+\nabla h(x)|^{\alpha_1}} + c_2\frac{z\nabla h(x) + \nabla h(x)^2}{|z+\nabla h(x)|^{\alpha_2}}\right).$$

We consider this equation evaluated on the set $\Omega_h = \{(x,z) \mid |\nabla h(x)| \geq |z|\}$. We first write the following inequality:

$$\dot{V}_2 \leq -c_1\nabla^2 h(x)\frac{|\nabla h(x)|^2 - |z||\nabla h(x)|}{|z+\nabla h(x)|^{\alpha_1}} - c_2\nabla^2 h(x)\frac{|\nabla h(x)|^2 - |z||\nabla h(x)|}{|z+\nabla h(x)|^{\alpha_2}}.$$

We readily see that $\dot{V}_2 \leq 0$ on Ω_h. Moreover, the second term on the right hand side is negative on Ω_h. Using the triangle inequality on $|z + \nabla h(x)|$, we get:

$$|z + \nabla h(x)| \leq |z| + |\nabla h(x)|$$

As a result, we can write the inequality as:

$$\dot{V}_2 \leq -c_1 \nabla^2 h(x) \frac{|\nabla h(x)|^2 - |z||\nabla h(x)|}{|z|^{\alpha_1} + |\nabla h(x)|^{\alpha_1}}$$

As above, we introduce the parameter $c \in (0, 1)$ and rewrite the last inequality:

$$\dot{V}_2 \leq -c_1 \nabla^2 h(x) \frac{(1-c)|\nabla h(x)|^2 + |\nabla h(x)|(c|\nabla h(x)| - |z|)}{|z|^{\alpha_1} + |\nabla h(x)|^{\alpha_1}}$$

On the set Ω_h, one can write that

$$|z|^{\alpha_1} + |\nabla h(x)|^{\alpha_1} \leq 2|\nabla h(x)|^{\alpha_1}.$$

Therefore $\dot{V}_2$ is negative definite for $|\nabla h(x)| \geq \frac{1}{c}|z|$. We define the set, $\Omega_h^c = \left\{(x, z) \;\middle|\; |\nabla h(x)| \geq \frac{1}{c}|z| \right\}$. Since $c \in (0, 1)$, one gets that $\Omega_h \subset \Omega_h^c$. Consequently, this yields:

$$\dot{V}_2 \leq -c_1 \nabla^2 h(x) \frac{(1-c)|\nabla h(x)|^2}{2|\nabla h(x)|^{\alpha_1}}$$

for all (x, z) such that $|\nabla h(x)| \geq \frac{1}{c}|z|$. Using the definition of V_2, and Assumption 1 we can write, as above:

$$\dot{V}_2 \leq -(1-c)\alpha_h \left(c_1 V_2^{1-\frac{\alpha_1}{2}} \right), \text{ if } V_2 \geq \frac{1}{c^2} V_1.$$

As a result, we conclude that the gradient dynamics are FTISS with z as an input.

Also, from the previous development, it can be deduced that V_1 satisfies the following inequality:

$$\dot{V}_1 \leq -(1-c)\alpha \left(c_1 V_1^{1-\frac{\alpha_1}{2}} + c_2 V_1^{1-\frac{\alpha_2}{2}} \right) = -\alpha_2(V_1), \text{ if } V_1 \geq \frac{1}{c^2\alpha^2} V_2.$$

Therefore we can view the finite-time systems as the interconnection of two FTISS nonlinear systems. We can apply the small gain theorem from Hong *et al*[15].

Following the analysis proposed in[21], we define the parameter σ chosen such that: $\frac{1}{c^2} \leq \sigma \leq c^2\alpha^2$. One then poses the following Lyapunov function candidate:

$$V(z, \nabla h) = \max \left\{ \sigma V_1(z),\; V_2(\nabla h) \right\}.$$

By the definitions of the functions V_1, V_2 and the constant σ, the function V is both proper and positive definite. It is also straightforward to see that the function $V(x, \nabla h)$ is locally Lipschitz and, therefore, differentiable almost everywhere. One can then follow the proof of Theorem 3.1 in[21] to show that:

$$D^+ V \leq -\min \left\{ \frac{1}{2}\alpha_1(V),\; \alpha_2(V) \right\}.$$

As a result, the system has a finite-time stable equilibrium at the optimum $z = 0$, $\nabla h(x) = 0$ for any $\alpha = K - \nabla^2 h(x) > 1$. Thus it follows that if the Hessian $\nabla^2 h(x)$ is globally bounded then there exists a K such that the system is globally finite-time (FT) stable. Otherwise, for any (arbitrary) compact set in the state space on which the Hessian is bounded, there exists a K such that the optimum is a semi-global finite-time stable equilibrium of the closed-loop system. This completes the proof. ■

Remark 1. It follows that if the Hessian is known then the gain K can be chosen as $K = \alpha + \nabla^2 h(x)$. In this case, we achieve finite-time stability for any $\alpha > 1$.

3.3 | Proposed Finite-time ESC

The proposed finite-time ESC approach is given by:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\gamma(\xi)\xi \\ \frac{\mathrm{d}\xi}{\mathrm{d}t} &= -\gamma(\xi - \delta)K(\xi - \delta). \end{aligned} \tag{14}$$

where $\delta = \dfrac{2}{a}h(x + a\sin(\omega t))\sin(\omega t)$. The right hand side of this time-varying nonlinear system is continuous with respect to x and t. As a result, we guarantee the existence of at least one Carathéodory solution which may not be unique.

A formal average of this system is given by:

$$\begin{aligned}\frac{\mathrm{d}x^{av}}{\mathrm{d}t} &= -\gamma(\xi^{av})\xi^{av}\\ \frac{\mathrm{d}\xi^{av}}{\mathrm{d}t} &= -\frac{K}{T}\int_0^T\left(\frac{c_1(\xi^{av}-\delta(x^{av},t))}{|\xi^{av}-\delta(x^{av},t)|^{\alpha_1}}+\frac{c_2(\xi^{av}-\delta(x^{av},t))}{|\xi^{av}-\delta(x^{av},t)|^{\alpha_2}}\right)dt,\ T>0\end{aligned} \tag{15}$$

The term $\xi - \delta(x,t)$ can be expanded in the following manner:

$$\begin{aligned}\xi - \delta(x,t) =& \xi - \frac{2}{a}h(x + a\sin(\omega t))\sin(\omega t)\\ =&\frac{1}{a}\left(a\xi - 2h(x)\sin(\omega t) - 2\nabla h(x)\sin^2(\omega t)a - a^2R(t,x,a,\omega)\right)\end{aligned}$$

where $R(t,x,a,\omega)$ is a function of higher order derivatives of $h(x)$, higher powers of the sinusoidal signals and the amplitude. This can be rewritten as:

$$\begin{aligned}\xi - \delta(x,t) =&\frac{1}{a}\left(-2h(x)\sin(\omega t) - \nabla h(x)(1-2\sin^2(\omega t))a + (\xi - \nabla h(x))a - a^2R(t,x,a,\omega)\right)\\ =&\frac{1}{a}\left(-2h(x)\sin(\omega t) - \nabla h(x)\sin(2\omega t)a + (\xi - \nabla h(x))a - a^2R(t,x,a,\omega)\right)\end{aligned}$$

Let us assume that the amplitude is picked small enough such that the last term is negligible:

$$\xi - \delta(x,t) \approx \frac{1}{a}\left(-2h(x)\sin(\omega t) - \nabla h(x)\sin(2\omega t)a + (\xi - \nabla h(x))a\right)$$

As a result, we obtain:

$$\frac{1}{T}\int_0^T(\xi - \delta(x,t))dt \approx (\xi - \nabla h(x)).$$

In addition, it is also easy to compute that:

$$\frac{1}{T}\int_0^T|\xi - \delta(x,t)|dt \approx |\xi - \nabla h(x)|.$$

Since there are no analytical expressions of the right hand side of (15), we cannot provide a suitable closed form expression for the resulting averaged system. In this study, we propose to consider the stability of the averaged system (15) directly. In the following, it is shown that the averaged system meets the stability conditions of the target averaged system presented in the previous section.

Proposition 1. Consider the nonlinear system (15). Let Assumption 1 be satisfied. Then the optimum $x = x^*$, $\xi = 0$ is a finite-time stable equilibrium of the system.

Proof: We consider the same change of coordinates to $z^{av} = \xi^{av} - \nabla h(x^{av})$ and $\nabla h(x^{av})$ and write the average dynamics as follows:

$$\begin{aligned}\frac{\mathrm{d}\nabla h(x^{av})}{\mathrm{d}t} &= -\nabla^2 h(x^{av})\gamma(z^{av}+\nabla h(x^{av}))(z^{av}+\nabla h(x^{av}))\\ \frac{\mathrm{d}z^{av}}{\mathrm{d}t} &= -\frac{K}{T}\int_0^T\left(\frac{c_1(\xi^{av}-\delta(x^{av},t))}{|\xi^{av}-\delta(x^{av},t)|^{\alpha_1}}+\frac{c_2(\xi^{av}-\delta(x^{av},t))}{|\xi^{av}-\delta(x^{av},t)|^{\alpha_2}}\right)dt + \nabla^2 h(x^{av})\gamma(z^{av}+\nabla h(x^{av}))(z^{av}+\nabla h(x^{av})).\end{aligned} \tag{16}$$

Then we pose the Lyapunov function candidates, $V_1^{av} = \frac{1}{2}(z^{av})^2$ and $V_2^{av} = \frac{1}{2}(\nabla h(x^{av}))^2$. The time derivative of V_1^{av} is given by:

$$\dot{V}_1^{av} = z^{av}\left(-\frac{K}{T}\int_0^T \left(\frac{c_1(\xi^{av} - \delta(x^{av},t))}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} + \frac{c_2(\xi^{av} - \delta(x^{av},t))}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_2}} \right) dt \right) + z^{av}\nabla^2 h(x^{av})\Big(K\gamma(z^{av} + \nabla h(x^{av}))(z^{av} + \nabla h(x^{av})) \Big) \tag{17}$$

If one substitutes for $z^{av} = \frac{1}{T}\int_0^T (\xi^{av} - \delta(x^{av},t))dt$, then the first term in the right hand side is clearly negative. However, we must confirm that the averaged system possesses the FTISS property demonstrated for the target system in the previous section.

We consider the first term in (17):

$$\Phi_1 = \frac{1}{T} z^{av} \int_0^T \left(\frac{c_1(\xi^{av} - \delta(x^{av},t))}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} \right) dt,$$

then

$$\Phi_1 = \left(\frac{1}{T}\int_0^T (\xi^{av} - \delta(x^{av},t))dt \right)\left(\frac{1}{T}\int_0^T \frac{c_1(\xi^{av} - \delta(x^{av},t))}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} dt \right).$$

Since the two terms have the same sign, this term is clearly nonnegative. As a result, it can be written as:

$$\Phi_1 = \left| \frac{1}{T}\int_0^T (\xi^{av} - \delta(x^{av},t))dt \right| \left| \frac{1}{T}\int_0^T \frac{c_1(\xi^{av} - \delta(x^{av},t))}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} dt \right|.$$

Similarly, we define

$$\Phi_2 = \left| \frac{1}{T}\int_0^T (\xi^{av} - \delta(x^{av},t))dt \right| \left| \frac{1}{T}\int_0^T \frac{c_2(\xi^{av} - \delta(x^{av},t))}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_2}} dt \right|.$$

Next we rewrite the following expression:

$$\left| \frac{1}{T}\int_0^T \frac{c_1(\xi^{av} - \delta(x^{av},t))}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} dt \right| \int_0^T |(\xi^{av} - \delta(x^{av},t))|^{\alpha_1} dt = \left| \frac{1}{T}\int_0^T c_1(\xi^{av} - \delta(x^{av},t)) \frac{\int_0^T |(\xi^{av} - \delta(x^{av},t))|^{\alpha_1} dt}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} dt \right|$$

It follows that for any t such that $(\xi^{av} - \delta(x^{av},t)) > 0$ then

$$(\xi^{av} - \delta(x^{av},t)) \frac{\int_0^T |(\xi^{av} - \delta(x^{av},t))|^{\alpha_1} dt}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} \geq (\xi^{av} - \delta(x^{av},t)).$$

Similarly, for any t such that $(\xi^{av} - \delta(x^{av},t)) < 0$, we have

$$(\xi^{av} - \delta(x^{av},t)) \frac{\int_0^T |(\xi^{av} - \delta(x^{av},t))|^{\alpha_1} dt}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} \leq (\xi^{av} - \delta(x^{av},t)).$$

As a result, it follows that:

$$\left| \frac{1}{T}\int_0^T c_1(\xi^{av} - \delta(x^{av},t)) \frac{\int_0^T |(\xi^{av} - \delta(x^{av},t))|^{\alpha_1} dt}{|\xi^{av} - \delta(x^{av},t)|^{\alpha_1}} dt \right| \geq \left| \frac{1}{T}\int_0^T c_1(\xi^{av} - \delta(x^{av},t)) dt \right|.$$

Simlarly, we obtain:

$$\left|\frac{1}{T}\int_0^T c_1(\xi^{av}-\delta(x^{av},t))\frac{\int_0^T|(\xi^{av}-\delta(x^{av},t))|^{\alpha_2}dt}{|\xi^{av}-\delta(x^{av},t)|^{\alpha_2}}dt\right| \geq \left|\frac{1}{T}\int_0^T c_1(\xi^{av}-\delta(x^{av},t)dt\right|.$$

Consequently, the last two inequalities can be used to show that:

$$\Phi_1\int_0^T|(\xi^{av}-\delta(x^{av},t))|^{\alpha_1}dt \geq c_1\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^2$$

and

$$\Phi_2\int_0^T|(\xi^{av}-\delta(x^{av},t))|^{\alpha_2}dt \geq c_2\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^2$$

This yields the following pair of inequalities:

$$\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|\left|\frac{1}{T}\int_0^T\frac{c_1(\xi^{av}-\delta(x^{av},t))}{|\xi^{av}-\delta(x^{av},t)|^{\alpha_1}}dt\right| \geq \frac{c_1\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^2}{\int_0^T|(\xi^{av}-\delta(x^{av},t))|^{\alpha_1}dt}$$

and:

$$\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|\left|\frac{1}{T}\int_0^T\frac{c_2(\xi^{av}-\delta(x^{av},t))}{|\xi^{av}-\delta(x^{av},t)|^{\alpha_2}}dt\right| \geq \frac{c_2\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^2}{\int_0^T|(\xi^{av}-\delta(x^{av},t))|^{\alpha_2}dt}$$

Finally, we apply Jensen's inequality on the denominator of the right hand side term in the last two equations and obtain the final inequalities:

$$\frac{c_1\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^2}{\int_0^T|(\xi^{av}-\delta(x^{av},t))|^{\alpha_1}dt} \geq \frac{c_1\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^2}{\left|\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^{\alpha_1}} = \frac{c_1(z^{av})^2}{|z^{av}|^{\alpha_1}},$$

and,

$$\frac{c_2\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^2}{\int_0^T|(\xi^{av}-\delta(x^{av},t))|^{\alpha_2}dt} \geq \frac{c_2\left|\frac{1}{T}\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^2}{\left|\int_0^T(\xi^{av}-\delta(x^{av},t))dt\right|^{\alpha_2}} = \frac{c_2(z^{av})^2}{|z^{av}|^{\alpha_2}}.$$

We can then substitute these inequalities in (18) as follows:

$$\dot{V}_1^{av} \leq -K\left(c_1\frac{|z^{av}|^2}{|z^{av}|^{\alpha_1}}+c_2\frac{|z^{av}|^2}{|z^{av}|^{\alpha_2}}\right)+z^{av}\nabla^2h(x^{av})\Big(K\gamma(z^{av}+\nabla h(x^{av}))(z^{av}+\nabla h(x^{av}))\Big) \tag{18}$$

Next we consider the dynamics of the gradient $\nabla h(x^{av})$ for the averaged system. As in the previous section, we consider the candidate Lyapunov function $V_2^{av} = \frac{1}{2}\nabla h(x^{av})^2$. Its time derivative is given by:

$$\dot{V}_2^{av} = -\nabla^2h(x^{av})\gamma(z^{av}+\nabla h(x^{av}))(z^{av}+\nabla h(x^{av}))\nabla h(x^{av}).$$

Repeating as in the proof of Theorem 3 in Section 3.2, we can use V_1^{av} and V_2^{av} to demonstrate that the averaged system is FT stable for any $K > K^*$. ■

Having established that the averaged system (15) achieves the performance of the proposed target system (9), we must prove that the trajectories of the ESC system (14) approach the trajectories of the averaged system. Since the right hand side of the dynamics are not Lipschitz, but only continuous, the application of standard averaging results that rely on Lipschitz properties is not suitable.

Many suitable averaging results have been proposed in the classical literature. In this study, we consider the classical Krasnosel'skii-Krein theorem[22] (generalized by Plotnikova for differential inclusions[23]) to demonstrate the closeness of solution of the nominal system and the averaged system over a compact set $D \subset \mathbb{R}^2$ as $a \to 0$.

The theorem can be stated as follows.

Theorem 4. [22] Consider the nonlinear system $\dot{X} = f(t, X, \epsilon)$ where,

1. the map $f(t, X, \epsilon)$ is continuous in t and X on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$,
2. there exists a positive constant $L > 0$ and a compact set $D \subset \mathbb{R}^n$ such that $\|f(t, X, \epsilon)\| \leq L$ for $t \in \mathbb{R}_{\geq 0}$, $X \in D$ and $\epsilon \in [0, \epsilon^*]$,
3. the averaged system

$$\dot{X}^{av} = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, X^{av}, 0)dt$$

exists with solutions defined on the set D.

Then, for any $\epsilon \leq \epsilon^*$, there exists constants δ and $\overline{T}$ such that:

$$\|X(t) - X^{av}\| \leq \delta$$

for $t \in [0, \overline{T}]$.

We can now state the final result of this study.

Theorem 5. Consider the ESC system (14). Let Assumption 1 be satisfied. Then there exists an a^* such that for all $a \in (0, a^*]$, the optimum $x = x^*$, $\xi = 0$ is a semi-globally practically finite-time stable equilibrium of system (14).

Proof: The proof proceeds in two steps. In the first step, we consider the application of Theorem 4. For the analysis of the proposed finite-time ESC, the Krasnosel'skii-Krein theorem can be applied as follows.

Consider the state, $X = [x - x^*, \xi]^T$, and the corresponding averaged variables $X^{av} = [x^{av} - x^*, \xi^{av}]^T$. Consider the system's dynamics:

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= -\gamma(\xi)\xi \\ \frac{d\xi}{dt} &= -\gamma(\xi - \delta(\tilde{x} + x^*, t))K(\xi - \delta(\tilde{x} + x^*, t)). \end{aligned} \tag{19}$$

and the corresponding average:

$$\begin{aligned} \frac{d\tilde{x}^{av}}{dt} &= -\gamma(\xi^{av})\xi^{av} \\ \frac{d\xi^{av}}{dt} &= -\frac{K}{T}\int_0^T \left(\frac{c_1(\xi^{av} - \delta(\tilde{x}^{av} + x^*, t))}{|\xi^{av} - \delta(\tilde{x}^{av} + x^*, t)|^{\alpha_1}} + \frac{c_2(\xi^{av} - \delta(\tilde{x}^{av} + x^*, t))}{|\xi^{av} - \delta(\tilde{x}^{av} + x^*, t)|^{\alpha_2}} \right) dt. \end{aligned} \tag{20}$$

By the analysis provided above, the averaged system has a finite-time stable equilibrium at the origin $X = 0$. Furthermore, the solutions of (20) exist and can be contained in a compact set $D \in \mathbb{R}^2$. Consider the nonlinear system (19). By the smoothness of the cost function $h(x)$ and the periodicity of the dither signal, it follows that the right hand side of the system can be bounded on a compact set $D \in \mathbb{R}^2$ uniformly in t. The continuity and the boundedness of the right hand side of (19) over a compact set D guarantees existence of solution of the averaged system. As a result, one can invoke the Krasnosel'skii-Krein theorem to guarantee that for any $a \in (0, a^*)$ there exists a $\overline{T}$ and a δ such that:

$$\|X(t) - X^{av}(t)\| \leq \delta$$

for $t \in [0, \overline{T}]$.

In the second step, we exploit the finite-time stability of the averaged system and the averaging result established in the first step to establish the finite-time practical semi-global stability of the ESC system.

Using the finite-time stability property of the averaged system (in particular, the corresponding generalized $\mathcal{K}_\infty$ function) and the averaging result for small amplitude signals, one can apply the approach in the proof of Theorem 1 in[24] to show that there exists a generalized class $\mathcal{K}_\infty$ function, β_X and a constant, c_X, such that:

$$\|X(t)\| \leq \beta_X(\|X(t_0)\|, t) + c_X$$

for $X(t_0) \in D$. This completes the proof. ∎

4 | SIMULATION STUDY

We consider the minimization of the cost function: $y = 1 + 2(x - 1)^2$. The finite extremum seeking controller is implemented with the following tuning parameters: $a = 1$, $q_1 = 3$, $q_2 = 1.5$, $c_1 = 1$, $c_2 = 1$, $k = 1$, $K = 250$ and $\omega = 300$. The initial conditions are: $x(0) = 3$ and $\xi(0) = 0.01$. The simulation results are shown in Figure 1. The top and middle plots of Figure 1 show the trajectories of the decision variable x and the auxiliary variable ξ along with the trajectories of the target averaged system. The bottom plot shows the corresponding value of the cost function for the finite-time ESC. The results demonstrate that the extremum seeking controller brings the system to the unknown optimum $y^* = 1$ in finite-time. The simulation results also demonstrate that the extremum seeking controller follows the trajectories of the target averaged system described in Section 3.

To emphasize the non-local nature of the finite-time convergence property of the finite-time ESC, we provide a simulation of the finite-time ESC with varying initial conditions, $x(0)$, ranging from -3 to 3. The corresponding trajectories of the system are show in Figure 2. As expected, the trajectories all converge to the correct optimum at the same finite-time in the interval $t \in [1.5, 2.0]$.

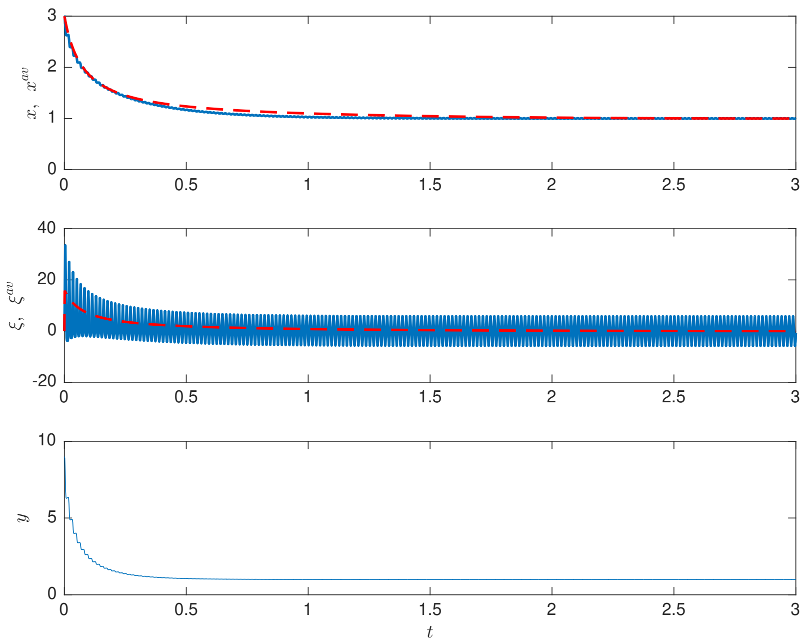

FIGURE 1 Performance of the Finite-time ESC. The graph shows the decision variable x (solid line) and its target averaged x^{av} (dashed line), the auxiliary variable ξ and the cost function y.

5 | CONCLUSION

In this study, we proposed an ESC design for the solution of real-time optimization problems for unknown static maps that achieves finite-time convergence to the unknown optimum. The proposed extremum seeking controller yields an averaged system with a finite-time stable equilibrium at the unknown optimum. Using classical averaging results for dynamical systems with continuous right hand sides, it is shown that the optimum is finite-time practically semi-globally stable equilibrium of the ESC system.

The class of static maps remains extremely restrictive when one considers practical situations. The proposed approach provides a firm theoretical foundation for the potential analysis of more complex systems. The first extension of the proposed technique will be to consider the development of Newton seeking techniques for multivariable problems. Newton seeking techniques can

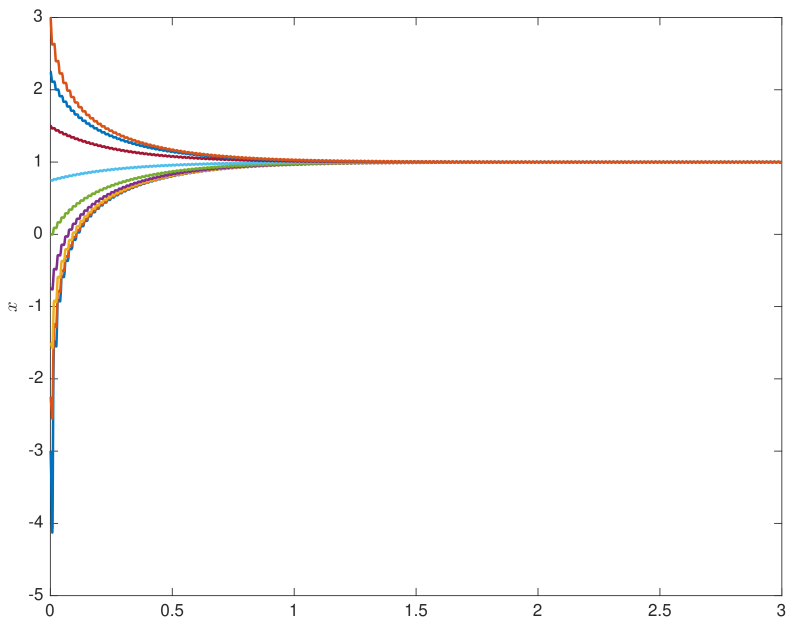

FIGURE 2 Performance of the Finite-time ESC. The graph shows the decision variable x (solid line) for the finite-time ESC with varying initial conditions.

be used to remove some important scaling issues arising in multivariable systems. Based on the methodology provided, it is very likely that a suitable Newton seeking technique can be developed. Another problem of interest will be the class of static maps subject to actuator limitations such as actuator delay, saturation and quantization.

Extension to dynamical systems requires a greater care since the practical finite-time stability of the system would require that the dynamical system provides some finite-time stability. In future work, we will consider the application of dual mode ESC techniques[25 26] for classes of dynamical systems that can achieve finite-time closed-loop stability.

Finally, extension of these methods to the case of discrete optimization algorithms, with accelerated convergence rates, remains a challenging open problem.

References

1. Tan Y., Moase W.H., Manzie C., Nesic D. and Mareels I.M.Y.. Extremum seeking from 1922 to 2010. In: *the 29th Chinese Control Conference (CCC)*:14 -26; 2010.

2. Krstic M., Wang H.H.. Stability of Extremum Seeking Feedback for General Dynamic Systems. *Automatica.* 2000;36(4):595-601.

3. Tan Y., Nesic D., Mareels I.. On non-local stability properties of extremum seeking control. *Automatica.* 2006;42(6):889 - 903.

4. Nagy Zoltán K, Braatz Richard D. Worst-case and distributional robustness analysis of finite-time control trajectories for nonlinear distributed parameter systems. *IEEE Transactions on Control Systems Technology.* 2003;11(5):694–704.

5. Ma David L, Braatz Richard D. Worst-case analysis of finite-time control policies. *IEEE Transactions on Control Systems Technology.* 2001;9(5):766–774.

6. Zhang Yanjiao, Yang Ying, Zhao Yu, Wen Guanghui. Distributed finite-time tracking control for nonlinear multi-agent systems subject to external disturbances. *International Journal of Control.* 2013;86(1):29-40.

7. Arabi E., Yucelen T., Singler J. R.. Robustness of Finite-Time Distributed Control Algorithm with Time Transformation. In: *IEEE American Control Conference (ACC)*:108-113; 2019.

8. Aghababa Mohammad Pourmahmood, Khanmohammadi Sohrab, Alizadeh Ghassem. Finite-time synchronization of two different chaotic systems with unknown parameters via sliding mode technique. *Applied Mathematical Modelling.* 2011;35(6):3080–3091.

9. Hui Qing. Finite-time rendezvous algorithms for mobile autonomous agents. *IEEE Transactions on Automatic Control.* 2010;56(1):207–211.

10. Bhat Sanjay P, Bernstein Dennis S. Finite-time stability of continuous autonomous systems. *SIAM Journal on Control and Optimization.* 2000;38(3):751–766.

11. Hong Yiguang. Finite-time stabilization and stabilizability of a class of controllable systems. *Systems & control letters.* 2002;46(4):231–236.

12. Hong Yigwruang, Huang Jie, Xu Yangsheng. On an output feedback finite-time stabilization problem. *IEEE Transactions on Automatic Control.* 2001;46(2):305–309.

13. Hong Yiguang, Yang Guowu, Bushnell Linda, Wang Hua O. Global finite-time stabilization: from state feedback to output feedback. In: *the 39th IEEE Conference on Decision and Control(CDC)* :2908–2913; 2000.

14. Hong Yiguang, Jiang Zhong-Ping. Finite-time stabilization of nonlinear systems with parametric and dynamic uncertainties. *IEEE Transactions on Automatic Control.* 2006;51(12):1950–1956.

15. Hong Yiguang., Jiang Zhong-Ping., Feng Gang. Finite-Time Input-to-State Stability and Applications to Finite-Time Control Design. *SIAM Journal on Control and Optimization.* 2010;48(7):4395-4418.

16. Polyakov Andrey, Nonlinear Feedback Design for Fixed-Time Stabilization of Linear Control Systems. *IEEE Transactions on Automatic Control.* 2012;57(8):2106-2110.

17. Francisco Lopez-Ramirez, Denis Efimov, Andrey Polyakov, Wilfrid Perruquetti. On Necessary and Sufficient Conditions for Fixed-Time Stability of Continuous Autonomous Systems. *the 17th European Control Conference (ECC). hal-01755079* 2018;.

18. Garg Kunal, Panagou Dimitra. New results on finite-time stability: Geometric conditions and finite-time controllers. In: *IEEE American Control Conference (ACC)* :442–447; 2018.

19. Romero Orlando, Benosman Mouhacine. Finite-Time Convergence of Continuous-Time Optimization Algorithms via Differential Inclusions. In: *the 33rd Conference on Neural Information Processing Systems (NeurIPS 2019), workshop: Beyond first order methods in machine learning systems*:1–11; 2019.

20. Romero Orlando, Benosman Mouhacine. Continuous-Time Optimization of Time-Varying Cost Functions via Finite-Time Stability with Pre-Defined Convergence Time. In: *the IEEE American Control Conference*: to appear; 2020.

21. Jiang Zhong-Ping, Mareels Iven MY, Wang Yuan. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica.* 1996;32(8):1211–1215.

22. Krasnosel'skii Mark Aleksandrovich, Krein Selim Grigor'evich. On the principle of averaging in nonlinear mechanics. *Uspekhi Matematicheskikh Nauk.* 1955;10(3):147–152.

23. Plotnikova Nataliya Viktorovna. The Krasnosel'skii-Krein theorem for differential inclusions. *Differential Equations.* 2005;41(7):1049–1053.

24. Teel A. R., Moreau L., Nesic D.. A unified framework for input-to-state stability in systems with two time scales. *IEEE Transactions on Automatic Control.* 2003;48(9):1526-1544.

25. Guay, M., Atta,K.T. . Dual mode extremum-seeking control via Lie-bracket averaging approximations. *2018 Annual American Control Conference (ACC), Milwaukee, WI*, June 27-29, 2018;2972–2977.

26. Guay, M. . Perturbation-Based Proportional Integral Extremum-Seeking Control Approach. *IEEE Transactions on Automatic Control* 2016; 61(11):3370–3381.

□