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# Robust Data-Driven Neuro-Adaptive Observers With Lipschitz Activation Functions

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*Abstract*—**While the use of neural networks for learning has gained traction in control and system identification problems, their use in data-driven estimator design is not as prevalent. Prior art on neuro-adaptive observers limit the type of activation functions to radial basis function networks and provide conservative bounds on the resulting observer estimation error because they leverage boundedness of the activation functions rather than exploiting their underlying structure. This paper proposes the use of Lipschitz activation functions in the neuro-adaptive observer: utilizing the Lipschitz constants of these activations simplifies the data-driven observer design procedure via recently discovered LMI conditions. Furthermore, in spite of measurement noise and approximation error, pre-computable robust stability guarantees are provided on the resulting state estimation error.**

*Index Terms*—**Data-driven; machine learning; neural networks; adaptive systems; nonlinear systems; linear matrix inequalities; function approximation.**

## I. Introduction

Function approximators such as neural networks have been widely used in control engineering. While the most common applications of neural nets are in identifying dynamical systems from measurement data [1], [2] or adapting with data to generate optimal control policies online [3]–[6], the utility of neural nets for state estimation in systems with unmodeled dynamics remains relatively unexplored.

Some early investigations into neuro-observers, for example, in [7] assume model availability. However, the current wave of data-driven control has demonstrated the effectiveness of approximators in controlling systems in a model-free manner. Neuro-observers in the model-free setting were explored almost two decades ago in [8], where the authors proposed an adaptation rule for learning the weights of a linear-in-parameter neural network (LPNN) that results in uniformly ultimately bounded estimation error dynamics. Although this work has been adopted in multiple applications such as robot control [9], [10], rotors [11], and more recently, wind turbines [12], the inherent assumptions and theory have hardly evolved. In most of these methodologies, the activation functions are considered to be radial basis functions, there is no measurement noise, and the theoretical guarantees of learning performance remain the same; an exception is [12] where the authors investigate input-to-state stability (ISS) observers for the known component of the model, but the learner performance is not ISS, and the learner's weights require extensive manual tuning.

This paper leverages recent work on observer design for nonlinear systems which are based on exploiting the structures of nonlinearities to formulate convex programs that can be solved to systematically generate observer gains [13]–[16]. The major contributions of this paper are as follows: (i) we propose the use of Lipschitz activation functions that are linear-in-parameter and formulate linear matrix inequalities, which, if solved, result in neuro-adaptive observers with guaranteed estimation performance with less conservative bounds than those presented in the literature; (ii) instead of adaptation gains obtained by backpropagation, we propose using adaptation laws as in [17] to incorporate robustness into data-driven observers design against measurement disturbances and approximation error. Our proposed observer is referred to as neuro-adaptive observer because the learner (i.e. adaptive element) is a neural network approximator.

The rest of the paper is organized as follows. The problem statement and proposed neuro-adaptive observer structure, along with assumptions made in the paper are provided in Section II. Sufficient conditions for guaranteeing the performance of the proposed observer and details of the observer design via linear matrix inequalities (LMIs) is provided in Section III.

*Notation:* We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ as the set of positive reals, and $\mathbb{N}$ as the set of natural numbers. For every $v \in \mathbb{R}^n$, we denote $\|v\| = \sqrt{v^\top v}$, where $v^\top$ is the transpose of $v$. The sup-norm is defined as $\|v\|_\infty \triangleq \sup_{t\in\mathbb{R}} \|v(t)\|$. We denote by $\underline{\sigma}(P)$ and $\overline{\sigma}(P)$ as the smallest and largest singular value of a square, symmetric matrix $P$, respectively. The $(i,j)$th element of $P$ is denoted $P_{ij} \equiv P_{i,j}$. The symbol $\succ$ ($\prec$) indicates positive (negative) definiteness and $A \succ B$ implies $A - B \succ 0$ for $A, B$ of appropriate dimensions. Similarly, $\succeq$ ($\preceq$) implies positive (negative) semi-definiteness. The operator norm is denoted $\|P\|$ and is defined as the maximum singular value of $P$, $\mathrm{vec}(P)$ denotes the column-wise vectorization of $P$, that is,

$$\mathrm{vec}\left(\begin{bmatrix} — & a_1^\top & — \\ — & a_2^\top & — \\ & \vdots & \\ — & a_n^\top & — \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

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and $\otimes$ denotes the Kronecker product. $I_p$ denotes a $p \times p$ identity matrix, $\mathbf{1}_p$ denotes a column vector of $p$ elements, all set to one, and $\delta_p(m)$ denotes a column vector of $p$ elements, where the $m$th element is one; $m \leq p$. The notation $\mathbf{0}_p$ denotes a column vector of $p$ zeros, and $\mathbf{0}_{p\times n}$ denotes a zero matrix of dimension $p\times n$; the symbol 0 without decorations may denote a matrix or scalar depending on the context.

## II. Problem Statement

### A. System description and assumptions

We consider nonlinear systems that have the state-space representation

$$\dot{x} = Ax + \varphi(x,u) \tag{1a}$$

$$y = Cx + Dv \tag{1b}$$

where $x \equiv x(t) \in \mathbb{R}^n$ represents the state of the system, $u \equiv u(t) \in \mathbb{R}^{n_u}$ represents the control input, $y \equiv y(t) \in \mathbb{R}^{n_y}$ represents the measured outputs, and $v \equiv v(t) \in \mathbb{R}^{n_v}$ denotes measurement noise. The nonlinearity $\varphi(x) : \mathbb{R}^n \to \mathbb{R}^n$ represents the unmodeled dynamics of the system and is unknown. The matrices $A$, $C$, and $D$ are constant with appropriate dimensions, and we require the pair $(A, C)$ to be observable.

We make the following assumptions on our system.

**Assumption 1.** *The state $x \in \mathcal{L}_\infty$ and the measurement noise $v \in \mathcal{L}_\infty$.*

Assumption 1 is satisfied, for example, when the system (1) is open-loop stable, which is a standard assumption in identification of dynamical systems. Since Assumption 1 implies that $x$ lies in a compact subset of $\mathbb{R}^n$, one can invoke neural network approximability results reported in [18] to represent the unknown nonlinearity as

$$\varphi(x,u) = \sum_{i=1}^{N} B_i W_i^\star + W^\star \sigma(x,u) + \varepsilon(x,u). \tag{2}$$

Here, $W^\star \in \mathbb{R}^{n\times N}$ is a constant but unknown matrix of weights in the output layer of a neural approximator with $N$ neurons, $W_i^\star$ is the $i$th column of $W^\star$, and $B_i \in \mathbb{R}^{n\times n}$ is a known matrix whose role is discussed in Remark 1. In addition, $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^N$ denotes the vector of activation functions, that is

$$\sigma(x,u) = \begin{bmatrix} \sigma_1(H_1 x, u) & \cdots & \sigma_N(H_N x, u) \end{bmatrix}^\top,$$

where $H_i \in \mathbb{R}^{s_i\times n}$. Note that the choice of the activation function $\sigma$ rests on the designer, and therefore, the exact form of $\sigma$ is completely known (although its argument $x$ is to be estimated). In the ensuing discussion, we will provide some guidelines for choosing $\sigma$ to ensure performance guarantees. The quantity $\varepsilon(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^n$ represents the neural network approximation error, which is bounded [18].

We make the following assumption on the neural approximator.

**Assumption 2.** *There exist positive scalars $\rho_W$, $\rho_\varepsilon$, and $\rho_\sigma$ such that*

$$\|W_\star\| \leq \rho_W, \ \|\varepsilon(\cdot,\cdot)\|_{\mathcal{L}_\infty} \leq \rho_\varepsilon, \ \textit{and} \ \|\sigma(\cdot,\cdot)\|_{\mathcal{L}_\infty} \leq \rho_\sigma.$$

*The activation functions are absolutely continuous on $\mathbb{R}^n$ and satisfy*

$$-\infty < \check{\sigma}_{ij} \leq \frac{\partial \sigma_i}{\partial q_j^i}(q_j, u) \leq \hat{\sigma}_{ij} \leq \infty, \tag{3}$$

*for every $q_j \in \mathbb{R}^{s_i}$ where the derivative exists, with $q_j = H_j x$, and $q_j^i$ being the $i$-th component of $q_j$.*

Traditional activation functions such as sigmoids and hyperbolic tangents satisfy Assumption 2. Modern activation functions that have gained traction in training deeper neural networks such as rectified linear units (ReLU) and their variants (Leaky ReLUs, scaled exponential linear units, exponential linear units) naturally satisfy the condition (3) except at zero, where they are typically non-differentiable, although their smoothened versions such as the softplus activation function satisfy these conditions everywhere. However, these activations are unbounded. In order to design data-driven observers using our proposed approach, one needs to ensure boundedness. A simple and common method to bound the activation function is through clipping. Clipped ReLU functions have demonstrated effectiveness in learning from time series data via deep nets [19].

**Remark 1.** As discussed in [17], one can assume $\check{\sigma}_{ij} = 0$ in (3) without loss of generality because the negative components can be translated using the linear component $\sum B_i W_i^\star$.

Replacing $\varphi(x,u)$ in (1) with the neural network basis representation (2), we get

$$\dot{x} = Ax + \sum_{i=1}^{N} B_i W_i^\star + W^\star \sigma(x,u) + \varepsilon(x,u). \tag{4}$$

### B. Proposed neuro-adaptive observer

In order to estimate the state of the system (1), we propose a data-driven neuro-adaptive observer of the form

$$\dot{\hat{x}} = A\hat{x} + \sum_{i=1}^{N} B_i \hat{W}_i + \hat{W}\bar{\sigma}(\hat{x},u) + L_0\big(y - C\hat{x}\big), \tag{5a}$$

$$\dot{\hat{W}}_{ij} = L_k\big(y - C\hat{x}\big), \tag{5b}$$

for $i = 1,\ldots,n$, $j = 1,\ldots,N$, and $k = 1,\ldots,nN$. Here,

$$\bar{\sigma}(\hat{x},u) = \begin{bmatrix} \sigma_1\left(H_1\hat{x} + K_1(y - C\hat{x}), u\right) \\ \vdots \\ \sigma_N\left(H_N\hat{x} + K_N(y - C\hat{x}), u\right) \end{bmatrix}, \tag{6}$$

$\hat{x} \equiv \hat{x}(t)$ is the estimated state of the plant (1), $\hat{W}$ denotes the vector of estimated weight parameters whose $(i,j)$th element is $\hat{W}_{ij}$, $L_0 \in \mathbb{R}^{n\times n_y}$, $L_k \in \mathbb{R}^{1\times n_y}$ for $k = 1,\ldots,nN$, and $K_i \in \mathbb{R}^{s_i\times n_y}$ for $i = 1,\ldots,N$ are observer gain matrices.

### C. Observer error dynamics

Let $\tilde{x} := x - \hat{x}$ denote the state estimation error and

$$\tilde{w} := \mathrm{vec}(W^\star) - \mathrm{vec}(\hat{W}) \in \mathbb{R}^{nN}$$

denote the parameter estimation error, and let the total disturbance input and the augmented error state be

$$d := \begin{bmatrix} \varepsilon \\ v \end{bmatrix} \text{ and } \tilde{z} := \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix}, \tag{7}$$

respectively. Therefore, $y - C\hat{x} = C\tilde{x} + Dv$.

Furthermore, we write the nonlinear term as

$$W^\star \sigma(x,u) = B_\psi \Psi(\vartheta, u)$$

and

$$\hat{W}\bar{\sigma}(x,u) = B_\psi \Psi(\hat{\vartheta}, u),$$

where

$$B_\psi = I_n \otimes \mathbf{1}_N^\top,\ \vartheta = \begin{bmatrix} x \\ \mathrm{vec}(W^\star) \end{bmatrix},\ \hat{\vartheta} = \begin{bmatrix} \hat{x} \\ \mathrm{vec}(\hat{W}) \end{bmatrix},$$

$$\Psi(\vartheta, u) = \begin{bmatrix} W_{11}^\star \sigma_1(H_1\, x, u) \\ \vdots \\ W_{1N}^\star \sigma_N(H_N\, x, u) \\ W_{k1}^\star \sigma_1(H_1\, x, u) \\ \vdots \\ W_{kN}^\star \sigma_N(H_N\, x, u) \\ \vdots \\ W_{n1}^\star \sigma_1(H_1\, x, u) \\ \vdots \\ W_{nN}^\star \sigma_N(H_N\, x, u) \end{bmatrix},$$

and

$$\Psi(\hat{\vartheta}, u) = \begin{bmatrix} \hat{W}_{11}\, \sigma_1(H_1\, \hat{x} + K_1 C\tilde{x} + K_1 Dv, u) \\ \vdots \\ \hat{W}_{1N}\, \sigma_N(H_N\, \hat{x} + K_N C\tilde{x} + K_N Dv, u) \\ \hat{W}_{k1}\, \sigma_1(H_1\, \hat{x} + K_1 C\tilde{x} + K_1 Dv, u) \\ \vdots \\ \hat{W}_{kN}\, \sigma_N(H_N\, \hat{x} + K_N C\tilde{x} + K_N Dv, u) \\ \vdots \\ \hat{W}_{n1}\, \sigma_1(H_1\, \hat{x} + K_1 C\tilde{x} + K_1 Dv, u) \\ \vdots \\ \hat{W}_{nN}\, \sigma_N(H_N\, \hat{x} + K_N C\tilde{x} + K_N Dv, u) \end{bmatrix}.$$

In the following result, we demonstrate a useful property of the nonlinearity $\Psi$.

**Lemma 1.** *Assumptions 1 and 2 hold. There exist functions $\psi_{ij} \triangleq \psi_{ij}(\vartheta, \hat{\vartheta}) : \mathbb{R}^{(N+1)n} \times \mathbb{R}^{(N+1)n} \to \mathbb{R}$, scalars $\check{\psi}_{ij}$ and $\hat{\psi}_{ij}$ such that*

$$\Psi(\hat{\vartheta}, u) - \Psi(\vartheta, u) = \sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \psi_{ij} \Delta_{ij} \big(\hat{\vartheta} - \vartheta\big)$$

*for any $\hat{\vartheta}, \vartheta \in \mathbb{R}^{(N+1)n}$, where*

$$\check{\psi}_{ij} \le \psi_{ij} \le \hat{\psi}_{ij} \quad \text{and} \quad \Delta_{ij} = \delta_{Nn}(i)\delta_{(N+1)n}^\top(j).$$

*Proof.* (Sketch) Without loss of generality, suppose $H_k = I_n$ and $K_k = 0$ for all $k = 1, \dots, N$. We begin by demonstrating that $\Psi$ is locally Lipschitz in its first argument on the compact subset of $\mathbb{R}^n$ where the neural approximation holds. To this end, consider

$$\begin{aligned} \|\Psi(\hat{\vartheta}, u) - \Psi(\vartheta, u)\| &= \left\| \sum_{k=1}^{Nn} \delta_{Nn}(k) \Big( \Psi_k(\hat{\vartheta}, u) - \Psi_k(\vartheta, u) \Big) \right\| \\ &\le \sum_{k=1}^{Nn} \|\Psi_k(\hat{\vartheta}, u) - \Psi_k(\vartheta, u)\| \\ &\le \sum_{k=1}^{Nn} \|w_k \sigma_k(x, u) - \hat{w}_k \sigma_k(\hat{x}, u)\|, \end{aligned}$$

with abuse of notation to avoid complicated index sets for the matrices $W^\star$ or $\hat{W}$. Note that $\Psi_k$ is the $k$-th component of $\Psi$.

Therefore, $\|\Psi(\hat{\vartheta}, u) - \Psi(\vartheta, u)\|$

$$\begin{aligned} &\le \sum_{k=1}^{Nn} \|w_k(\sigma_k(x,u) - \sigma_k(\hat{x}, u)) + (w_k - \hat{w}_k)\sigma_k(\hat{x}, u)\| \\ &\le \sum_{k=1}^{Nn} \rho_W \mathfrak{L}_\sigma \|x - \hat{x}\| + \rho_\sigma \|w_k - \hat{w}_k\|. \end{aligned}$$

Due to the equivalence of norms in compact subsets of finite-dimensional vector spaces, and after some developments, the final inequality implies that there is a scalar $\mathfrak{L}_\Psi$ large enough, depending on $n, N, \rho_W$, and $\rho_\sigma$, to satisfy

$$\|\Psi(\hat{\vartheta}, u) - \Psi(\vartheta, u)\| \le \mathfrak{L}_\Psi \|\hat{\vartheta} - \vartheta\|,$$

which affirms $\Psi$ is Lipschitz. Using the result of [15, Lemma 2], we conclude the proof. □

As in (3), without loss of generality, we assume that Lemma 1 gives $\check{\psi}_{ij} = 0$. Otherwise, we proceed as in [17, Remark 1].

Based on the abvoe discussion, the augmented error dynamics of the neural observer are given by

$$\dot{\tilde{z}} = (A_z - LC_z)\tilde{z} + B_z \Delta\Psi + (B_d - LD_d)d, \tag{8}$$

where

$$A_z = \begin{bmatrix} A & B_1 & \dots & B_N \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad L = \begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_{Nn} \end{bmatrix},$$

$$B_z = \begin{bmatrix} B_\psi \\ \mathbf{0}_{Nn \times Nn} \end{bmatrix}, \qquad B_d = \begin{bmatrix} I_n & \mathbf{0}_{n \times n_v} \\ \mathbf{0}_{nN \times n} & \mathbf{0}_{nN \times n_v} \end{bmatrix},$$

$$C_z = \begin{bmatrix} C & \mathbf{0}_{Nn}^\top \end{bmatrix}, \qquad D_d = \begin{bmatrix} \mathbf{0}_n^\top & D \end{bmatrix},$$

and

$$\begin{aligned} \Delta\Psi &= \Psi(\vartheta, u) - \Psi(\hat{\vartheta}, u) \\ &= \sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \psi_{ij}(\vartheta, \hat{\vartheta}) \Delta_{ij}\, (\bar{H}_i \tilde{z} - \bar{K}_i D_d d), \end{aligned} \tag{9}$$

where

$$\bar{K}_i = \begin{bmatrix} K_i \\ \mathbf{0}_{Nn\times n_y} \end{bmatrix}$$

and

$$\bar{H}_i = \begin{bmatrix} H_i & \mathbf{0}_{s_i\times Nn} \\ \mathbf{0}_n^\top & \delta_{Nn}^\top(i) \end{bmatrix} - \bar{K}_i C_z.$$

### D. Objective of the paper

To formally state our objective, we require the following definition from [14].

Let $\mu$ be a non-negative scalar and

$$y_z = F_z \tilde{z} \tag{10}$$

be a performance output of the error system (8).

**Definition 1.** The input-output system (8) with performance output (10) is globally uniformly $\mathcal{L}_\infty$-stable with performance level $\mu$ if it has the following properties.

(P1) *Global uniform exponential stability with zero input.* The zero-input system ($d \equiv 0$) is globally uniformly exponentially stable about the origin.

(P2) *Global uniform boundedness of the error state.* For every initial condition $\tilde{z}(t_0) = \tilde{z}_0$, and every bounded exogenous input $d(\cdot)$, there exists a non-negative upper bound $\beta(\tilde{z}_0, \|d(\cdot)\|_\infty)$ such that

$$\|\tilde{z}(t)\| \le \beta(\tilde{z}_0, \|d(\cdot)\|_\infty)$$

for all $t \ge t_0$.

(P3) *Output response for zero initial error state.* For zero initial error, $\delta(t_0) = 0$, and every bounded exogenous input $d(\cdot)$, we have

$$\|y_z(t)\| \le \mu\|d(\cdot)\|_\infty$$

for all $t \ge t_0$.

(P4) *Global ultimate output response.* For every initial condition, $\tilde{z}(t_0) = \tilde{z}_0$, and every bounded exogenous input $d(\cdot)$, we have

$$\limsup_{t\to\infty} \|y_z(t)\| \le \mu\|d(\cdot)\|_\infty.$$

Moreover, convergence is uniform with respect to $t_0$.

Our objective is to design gains $\{L_i\}_{i=0}^{nN}$ and $\{K_i\}_{i=1}^{N}$ of the neural observer such that the augmented error system is $\mathcal{L}_\infty$-stable with a prescribed performance level $\mu$.

## III. Performance Analysis of the Data-Driven Observer

### A. Sufficient conditions for $\mathcal{L}_\infty$-stability

The following lemma describes conditions requires to guarantee that the augmented error system (8) is $\mathcal{L}_\infty$-stable with a performance level $\mu$.

**Lemma 2** ([14]). *If there exists a matrix $P = P^\top \succ 0$ and scalars $\beta, \mu > 0$ such that*

$$\dot{V} + 2\beta V - 2\beta\|d\|^2 \le 0 \tag{11a}$$

$$\|F_z\tilde{z}\|^2 \le \mu^2 V, \tag{11b}$$

*where $V = \tilde{z}^\top P\tilde{z}$, then the augmented error system (8) is $\mathcal{L}_\infty$-stable with performance level $\mu$, with $\beta$ denoting the exponential decay term.*

### B. A design theorem

The following result enables us to design the neural observer using convex programming by solving a set of linear matrix inequalities.

**Theorem 1.** *For a non-negative scalar $\mu$, if there exist matrices $P = P^\top \succ 0$, $R$, $K_i$ of appropriate dimensions and a fixed scalar $\beta > 0$ such that*

$$\Xi \preceq 0, \tag{12a}$$

$$\begin{bmatrix} P & \star \\ F_z & \mu^2 I \end{bmatrix} \succeq 0, \tag{12b}$$

*where*

$$\Xi := \begin{bmatrix} \Xi_{11} & PB_d - RD_d & PB_z\Sigma_1 + \Sigma_2 & -PB_z\Sigma_1 \\ \star & -2\beta I_{n+n_v} & 0 & \Sigma_3 \\ \star & \star & -\Lambda & 0 \\ \star & \star & \star & -\Lambda \end{bmatrix}$$

*with*

$$\Xi_{11} = A_z^\top P + PA_z - RC_z - C_z^\top R^\top + 2\beta P$$
$$\Sigma_1 = \begin{bmatrix} \Delta_{11} & \cdots & \Delta_{1,n(N+1)} & \Delta_{21} & \cdots \Delta_{Nn,(N+1)n} \end{bmatrix},$$
$$\Sigma_2 = \begin{bmatrix} \mathbf{1}_{(N+1)n}^\top \otimes \bar{H}_1^\top & \cdots & \mathbf{1}_{(N+1)n}^\top \otimes \bar{H}_{Nn}^\top \end{bmatrix},$$
$$\Sigma_3 = \begin{bmatrix} \mathbf{1}_{(N+1)n}^\top \otimes (\bar{K}_1 D_d)^\top & \cdots & \mathbf{1}_{(N+1)n}^\top \otimes (\bar{K}_{Nn} D_d)^\top \end{bmatrix},$$
$$\Lambda = \operatorname{diag}\left(\begin{bmatrix} \frac{2}{\hat{\psi}_{11}} & \cdots & \frac{2}{\hat{\psi}_{1,(N+1)n}} & \frac{2}{\hat{\psi}_{21}} & \cdots & \frac{2}{\hat{\psi}_{Nn,(N+1)n}} \end{bmatrix}\right),$$

*then the augmented error system (8) with observer gains $L = P^{-1}R$ is $\mathcal{L}_\infty$-stable with performance level $\mu$.*

*Proof.* Let $V = \tilde{z}^\top P\tilde{z}$. Taking the time derivative of $V$ along the trajectories of the augmented error dynamics, we get

$$\begin{aligned}\dot{V} &= \tilde{z}^\top((A_z - LC_z)^\top P + P(A_z - LC_z)\tilde{z} \\ &\quad + 2\tilde{z}^\top P(B_d - LD_d)d + 2\tilde{z}^\top PB_z\Delta\Psi.\end{aligned}$$

Exploiting the structure of the nonlinearity via (9) yields

$$\begin{aligned}\dot{V} &= \tilde{z}^\top((A_z - LC_z)^\top P + P(A_z - LC_z)\tilde{z} \\ &\quad + 2\tilde{z}^\top P(B_d - LD_d)d \\ &\quad + 2\sum_{i=1}^{Nn}\sum_{j=1}^{(N+1)n} \psi_{ij}\tilde{z}^\top PB_z\Delta_{ij}\bar{H}_i\tilde{z} \\ &\quad - 2\sum_{i=1}^{Nn}\sum_{j=1}^{(N+1)n} \psi_{ij}\tilde{z}^\top PB_z\Delta_{ij}\bar{K}_i D_d\, d \qquad (13) \\ &= \tilde{z}^\top(\Xi_{11} - I)\tilde{z} + 2\tilde{z}^\top(PB_d - RD_d)d \\ &\quad + 2\tilde{z}^\top(PB_z\Sigma_1)\zeta_1 + 2\tilde{z}^\top(PB_z\Sigma_1)\zeta_2, \qquad (14)\end{aligned}$$

where

$$\zeta_1 = \begin{bmatrix} \psi_{11}\bar{H}_1\tilde{z} \\ \vdots \\ \psi_{1,N(n+1)}\bar{H}_1\tilde{z} \\ \psi_{21}\bar{H}_2\tilde{z} \\ \vdots \\ \psi_{Nn,(N+1)n}\bar{H}_{Nn}\tilde{z} \end{bmatrix} \text{ and } \zeta_2 = \begin{bmatrix} \psi_{11}\bar{K}_1 D_d\, d \\ \vdots \\ \psi_{1,N(n+1)}\bar{K}_1 D_d\, d \\ \psi_{21}\bar{K}_2 D_d\, d \\ \vdots \\ \psi_{Nn,(N+1)n}\bar{K}_{Nn} D_d\, d \end{bmatrix}.$$

Let

$$\Gamma = \begin{bmatrix} \tilde{z}^\top & d^\top & \zeta_1^\top & \zeta_2^\top \end{bmatrix}^\top. \tag{15}$$

Taking a congruence transformation of (12a) with $\Gamma$ and replacing (14) yields

$$0 \geq \Gamma^\top \Xi \Gamma = \dot{V} + 2\beta V - 2\beta \|d\|^2 + \zeta, \tag{16}$$

where

$$\begin{aligned}
\zeta &= 2\tilde{z}^\top \Sigma_2 \zeta_1 + 2d^\top \Sigma_3 \zeta_2 + \sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \frac{2}{\hat{\psi}_{ij}} \zeta_1^\top \zeta_1 + \frac{2}{\hat{\psi}_{ij}} \zeta_2^\top \zeta_2 \\
&= 2\sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \left( (\bar{H}_i \tilde{z})^\top \zeta_1 - \frac{1}{\hat{\psi}_{ij}} \zeta_1^\top \zeta_1 \right) \\
&\quad + 2\sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \left( (\bar{K}_i d)^\top \zeta_2 - \frac{1}{\hat{\psi}_{ij}} \zeta_2^\top \zeta_2 \right) \\
&= 2\sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \left( \frac{1}{\psi_{ij}} - \frac{1}{\hat{\psi}_{ij}} \right) \|\zeta_1\|^2 \\
&\quad + 2\sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \left( \frac{1}{\psi_{ij}} - \frac{1}{\hat{\psi}_{ij}} \right) \|\zeta_2\|^2,
\end{aligned} \tag{17}$$

because

$$\sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \bar{H}_i \tilde{z} = \sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \frac{1}{\psi_{ij}} \zeta_1$$

and

$$\sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \bar{K}_i d = \sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \frac{1}{\psi_{ij}} \zeta_2.$$

Since $\psi_{ij} \leq \hat{\psi}_{ij}$ by construction, the right hand side of (17) is a finite sum of non-negative elements. Therefore, $\zeta \geq 0$. Plugging this into (16) yields the inequality (11a).

Taking Schur complements of (12b) and recalling (10) yields (11b). As both conditions of Lemma 2 are satisfied, we invoke the lemma and conclude the proof. ☐

**Remark 2.** One can improve the solution quality by solving the problem $\min(\mu^2)$ subject to (12a)-(12b). This will result in the observer gains that reduce the effect of the disturbance input $d$ by reducing $\mu$.

The term $-PB_z\Sigma_1$ in (12a) forces $\hat{\psi}_{ij}$ to be finite. Some extra conditions are required in order to ensure that the observer error dynamics are $\mathcal{L}_\infty$-stable for nonlinearities with unbounded derivatives. These conditions are presented in the following corollary.

**Corollary 1.** *Suppose that $\hat{\psi}_{ij} = \infty$ for some $i$ in the index set $\mathcal{I} \subset \{1, \ldots, nN\}$. If the measurement noise $v = 0$ and there exist matrices $P = P^\top \succ 0$, $R$, $K_i$ of appropriate dimensions and a fixed scalar $\beta > 0$ such that*

$$\begin{bmatrix} \Xi_{11} & PB_d & PB_z\Sigma_1 + \Sigma_2 \\ \star & -2\beta I_n & 0 \\ \star & \star & -\Lambda \end{bmatrix} \preceq 0, \tag{18a}$$

$$PB_z\Delta_{ij} + \bar{H}_i = 0 \qquad \forall i \in \mathcal{I}. \tag{18b}$$

*and* (12b) *for non-negative $\mu$, then the augmented error system* (8) *with observer gains $L = P^{-1}R$ is $\mathcal{L}_\infty$-stable with performance level $\mu$.*

*Proof.* Since $v = 0$, the corresponding matrices $D_w$, $\Sigma_3$, and $\zeta_2$ vanish. The proof has the same ideas as Theorem 1, with the exception that

$$\zeta = \sum_{i \in \mathcal{I}} \sum_{j=1}^{(N+1)n} \left( \frac{1}{\psi_{ij}} - \frac{1}{\hat{\psi}_{ij}} \right) \|\zeta_1\|^2,$$

since the other terms vanish due to (18b). ☐

**Remark 3.** In the absence of $v$, note that the performance is quantified with respect to the bound on the disturbance input $\varepsilon$. Previous guarantees provided in neuro-adaptive observer design with no measurement noise [8]–[10], [12] consider the disturbance input $W^\star(\sigma - \hat{\sigma}) + \varepsilon$ and performance bounds are computed based on this disturbance. Since $\|W^\star(\sigma - \hat{\sigma}) + \varepsilon\|_\infty > \rho_\varepsilon$ with non-zero activation functions and weights, our design is less conservative than the prior art.

### C. Improved LMI conditions

Some recent work has exploited a variant of Young's inequality and linear parameter-varying systems theory to improve the LMI conditions proposed above. To this end, we present the following result [13].

**Lemma 3** ([13])**.** *Let $X$ and $Y$ denote matrices of appropriate dimensions such that $X^\top Y$ and $Y^\top X$ can be computed. Then, for any scalar $\alpha \in (0, 1)$ and symmetric matrices $S \succ 0$ and $Z \succ 0$ of appropriate dimensions, the following inequality holds:*

$$\begin{aligned} X^\top Y + Y^\top X \preceq \frac{(1-\alpha)}{2} \Big(X + SY\Big)^\top S^{-1} \Big(X + SY\Big) \\ + \frac{\alpha}{2} \Big(X + ZY\Big)^\top Z^{-1} \Big(X + ZY\Big). \end{aligned} \tag{19}$$

Our next result enables the computation of observer gains using improved LMI conditions.

**Theorem 2.** *For a non-negative scalar $\mu$, if there exist matrices $P = P^\top \succ 0$, $S_{ij} = S_{ij}^\top \succ 0$, $Q_{ij} = Q_{ij}^\top \succ 0$, $R$, $K_i$ of appropriate dimensions and fixed scalars $\beta > 0$, $\pi_{ij} \in (0, 1)$ for $1 \leq i \leq Nn$ and $1 \leq j \leq (N+1)n$, such that* (12b) *and*

$$\begin{bmatrix} \Xi_{11} & PB_d - RD_d & \mathcal{N} & 0 \\ \star & -2\beta I_{n+n_v} & 0 & -\mathcal{N}' \\ \star & \star & -\Lambda_1 & 0 \\ \star & \star & \star & -\Lambda_1 \end{bmatrix} \preceq 0, \tag{20}$$

*where*

$$\begin{aligned} \mathcal{N} &= \begin{bmatrix} \mathcal{N}_{11} & \cdots & \mathcal{N}_{ij} & \cdots & \mathcal{N}_{Nn,(N+1)n} \end{bmatrix} \\ \mathcal{N}' &= \begin{bmatrix} \mathcal{N}'_{11} & \cdots & \mathcal{N}'_{ij} & \cdots & \mathcal{N}'_{Nn,(N+1)n} \end{bmatrix} \end{aligned}$$

*with*

$$\begin{aligned} \mathcal{N}_{ij} &= \big[PB_z\Delta_{ij} + (\bar{H}_i + \bar{K}_i C_z)S_{ij}, \\ &\qquad PB_z\Delta_{ij} + (\bar{H}_i + \bar{K}_i C_z)Q_{ij}\big], \\ \mathcal{N}'_{ij} &= \big[PB_z\Delta_{ij} + \bar{K}_i D_d S_{ij} \quad PB_z\Delta_{ij} + \bar{K}_i D_d Q_{ij}\big], \end{aligned}$$

*and*

$$\Lambda_1 = \mathrm{blkdiag}\begin{bmatrix} \frac{2}{(1-\pi_{11})\hat{\psi}_{11}} S_{11} \\ \frac{2}{(1-\pi_{11})\hat{\psi}_{11}} Q_{11} \\ \vdots \\ \frac{2}{(1-\pi_{ij})\hat{\psi}_{ij}} S_{ij} \\ \frac{2}{(1-\pi_{ij})\hat{\psi}_{ij}} Q_{ij} \\ \vdots \end{bmatrix}$$

*then the augmented error system* (8) *with observer gains* $L = P^{-1}R$ *is* $\mathcal{L}_\infty$*-stable with performance level* $\mu$.

*Sketch of proof.* We proceed as in the proof of Theorem 1 with $V = \tilde{z}^\top P \tilde{z}$, and compute $\dot{V}$ to be (13).

From the definition of $\Lambda_1$, it is clear that $\Lambda_1 \succ 0$. Then taking Schur complements of (20) yields

$$\begin{bmatrix} \Xi_{11} & PB_d - RD_d \\ \star & -2\beta I \end{bmatrix} + \underbrace{\begin{bmatrix} \mathcal{N} & 0 \\ 0 & \mathcal{N}' \end{bmatrix}^\top \begin{bmatrix} \Lambda_1^{-1} & 0 \\ 0 & \Lambda_1^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{N} & 0 \\ 0 & \mathcal{N}' \end{bmatrix}}_{T_2} \preceq 0. \quad (21)$$

Using Lemma 3 on the second term $T_2$ implies

$$T_2 \succeq \mathrm{blkdiag}\begin{bmatrix} \sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \hat{\psi}_{ij} PB_z \Delta_{ij}(\bar{H}_i + \bar{K}_i C_z) \\ \sum_{i=1}^{Nn} \sum_{j=1}^{(N+1)n} \hat{\psi}_{ij} PB_z \Delta_{ij} \bar{K}_i D_d \end{bmatrix}. \quad (22)$$

Replacing $T_2$ in (21) with the right hand side of (22) and taking a congruence transform of this inequality with the vector $\begin{bmatrix} \tilde{z}^\top & d^\top \end{bmatrix}^\top$, we get

$$\dot{V} + 2\beta V - 2\beta \|d\|^2 \leq 0,$$

using similar arguments to the proof of Theorem 1. □

**Remark 4.** As demonstrated in [13], the number of decision variables can be considerably reduced by selecting a subset of $\{1, \cdots, Nn\} \times \{1, \cdots, (N+1)n\}$ where the $\hat{\psi}_{ij}$ value is above a pre-selected threshold, and solving the LMIs (2) for that smaller subset.

## IV. Conclusions

This paper provides a systematic design methodology to construct data-driven observers via neural approximators with bounded Lipschitz activation functions. We extend the current literature to incorporate $\mathcal{L}_\infty$-stability in the neural observer error dynamics by computing observer gains and adaptation weights via convex programming. Major advantages of our method include providing robustness to the approximation error and measurement noise explicitly in the observer design methodology, as well as learning the nonlinearity that can be used as part of a predictive model for subsequent control systems.

## References


[1] K. S. Narendra and K. Parthasarathy, "Identification and control of dynamical systems using neural networks," *IEEE Transactions on Neural Networks*, vol. 1, no. 1, pp. 4–27, 1990.

[2] E. B. Kosmatopoulos, M. M. Polycarpou, M. A. Christodoulou, and P. A. Ioannou, "High-order neural network structures for identification of dynamical systems," *IEEE Transactions on Neural Networks*, vol. 6, no. 2, pp. 422–431, 1995.

[3] B. Kiumarsi, K. G. Vamvoudakis, H. Modares, and F. L. Lewis, "Optimal and autonomous control using reinforcement learning: A survey," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 6, pp. 2042–2062, 2018.

[4] Y. Huang and D. Liu, "Neural-network-based optimal tracking control scheme for a class of unknown discrete-time nonlinear systems using iterative adp algorithm," *Neurocomputing*, vol. 125, pp. 46–56, 2014.

[5] A. Chakrabarty, D. K. Jha, G. T. Buzzard, Y. Wang, and K. Vamvoudakis, "Safe Approximate Dynamic Programming Via Kernelized Lipschitz Estimation," pp. 1–13, Jul 2019. [Online]. Available: http://arxiv.org/abs/1907.02151

[6] A. Chakrabarty, R. Quirynen, C. Danielson, and W. Gao, "Approximate dynamic programming for linear systems with state and input constraints," in *2019 18th European Control Conference (ECC)*, June 2019, pp. 524–529.

[7] J. Theocharis and V. Petridis, "Neural network observer for induction motor control," *IEEE Control Systems Magazine*, vol. 14, no. 2, pp. 26–37, 1994.

[8] Y. H. Kim, F. L. Lewis, and C. T. Abdallah, "A dynamic recurrent neural-network-based adaptive observer for a class of nonlinear systems," *Automatica*, vol. 33, no. 8, pp. 1539–1543, 1997.

[9] F. Abdollahi, H. Talebi, and R. Patel, "A Stable Neural Network-Based Observer With Application to Flexible-Joint Manipulators," *IEEE Transactions on Neural Networks*, vol. 17, no. 1, pp. 118–129, jan 2006.

[10] X. Liu, C. Yang, Z. Chen, M. Wang, and C. Y. Su, "Neuro-adaptive observer based control of flexible joint robot," *Neurocomputing*, vol. 275, pp. 73–82, 2018.

[11] B. Pratap and S. Purwar, "Neural Network Observer for Twin Rotor MIMO System: An LMI Based Approach," in *International Conference on Modeling, Identification and Control*. Okayama, Japan: IEEE, 2010, pp. 539–544.

[12] R. Rahimilarki, Z. Gao, A. Zhang, and R. J. Binns, "Robust neural network fault estimation approach for nonlinear dynamic systems with applications to wind turbine systems," *IEEE Transactions on Industrial Informatics*, vol. 3203, no. c, pp. 1–1, 2019.

[13] A. Zemouche, R. Rajamani, H. Trinh, and M. Zasadzinski, "A new LMI based $\mathcal{H}_\infty$ observer design method for Lipschitz nonlinear systems," in *Proc. European Control Conference (ECC)*, Jun 2016, pp. 2011–2016.

[14] A. Chakrabarty, M. J. Corless, G. T. Buzzard, S. H. Żak, and A. E. Rundell, "State and unknown input observers for nonlinear systems with bounded exogenous inputs," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5497–5510, 2017.

[15] A. Zemouche and M. Boutayeb, "On LMI conditions to design observers for Lipschitz nonlinear systems," *Automatica*, vol. 49, no. 2, pp. 585–591, 2013.

[16] A. Chakrabarty, E. Fridman, S. H. Żak, and G. T. Buzzard, "State and unknown input observers for nonlinear systems with delayed measurements," *Automatica*, vol. 95, pp. 246–253, sep 2018.

[17] A. Zemouche and M. Boutayeb, "A unified $\mathcal{H}_\infty$ adaptive observer synthesis method for a class of systems with both lipschitz and monotone nonlinearities," *Systems & Control Letters*, vol. 58, no. 4, pp. 282–288, 2009.

[18] K. Hornik, "Approximation capabilities of multilayer feedforward networks," *Neural Networks*, vol. 4, no. 2, pp. 251–257, 1991.

[19] A. Hannun, C. Case *et al.*, "Deep speech: Scaling up end-to-end speech recognition," *arXiv preprint arXiv:1412.5567*, 2014.