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Accelerating Convergence to Competitive Equilibrium in Electricity Markets

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***Abstract*—We present a single time-period decentralized market clearing model based on the DC power flow model. The electricity market we study consists of a set of Generation Companies (GenCos) and a set of Distribution System Operators (DSOs). We model the DSOs as a single node having deferrable loads. The Independent System Operator (ISO) determines the market clearing generation and demand levels by coordinating with the market participants (GenCos and DSOs). It is assumed that each market participant shares limited information with the ISO. We exploit the problem structure to obtain a decomposition of the market-clearing problem where the GenCos and DSOs are decoupled. We propose a novel semismooth Newton algorithm to compute the competitive equilibrium. Numerical experiments demonstrate that the algorithm can obtain several orders of magnitude speedup over a typical subgradient algorithm.**

I. Introduction

Electricity markets are commodity markets where: (i) suppliers (electricity generators) and consumers (electricity customers) are spread across a network and (ii) the flow of the commodity (electricity) is dictated by physical laws [1]. Equilibrium in an electricity market refers to a condition where a market price is established through competition. The design of appropriate market or pricing mechanisms is governed by the theory of general equilibrium. For example, competition [2] and active participation (e.g., demand response) [3] in these markets are known to significantly enhance efficiency and reduce prices. Given the importance of an efficient and reliable grid infrastructure, the modeling and subsequent analysis of electricity markets has seen extensive research. Using the DC network flow model for power flow: Hobbs and Helman [4] study market equilibrium via competitive equilibrium models; Hobbs, Metzler and Pang [5] study oligopolistic price equilibrium using supply functions; Baldick [6] compares Cournot and supply function equilibrium models of bid-based electricity markets; Weber and Overbye [7] study Nash equilibria for electricity markets; and Wang et al. [8] study competitive equilibria in dynamic electricity markets. The DC power flow model is not valid when voltage or reactive power constraints are considered. Motto et al [9] and Lavaei and Sojuodi [10] investigate market equilibrium for AC power flow networks.

A. Our Focus

While research has focused largely on aspects of electricity market design, there has been little work on algorithmic and computational aspects. This is especially important in the current context of grid infrastructure modernization and increased penetration of distributed generation. ARPA-E envisions that the future grid infrastructure will be able to incorporate diverse distributed generation sources with storage and operation under a distributed architecture for control [11]. In that context, it is important to develop decentralized or distributed algorithms that scale with network size and have little overhead in communication. This serves as the motivation and focus of this paper.

B. Our Contribution

We consider a pool-based electricity market consisting of: generation companies (GenCo), load entities called Distribution System Operators (DSO) and an Independent System Operator (ISO). We assume that: (a) the DC power flow model is used by the ISO to model the power flow in the transmission system, (b) the DSOs are modeled as a single node neglecting the underlying distribution network, (c) the DSOs have the ability to defer loads and (d) the GenCos and DSOs are price-taking and unwilling to share their cost function to the ISO. Maintaining privacy of the individual market agents motivates the development of a decentralized framework whereby the ISO only transmits price signals to the individual agents and obtains price-sensitive optimal actions from them. Using such obtained information, a subgradient algorithm [12] is typically employed by the ISO to obtain convergence to equilibrium. The convergence rate for subgradient algorithms is known to be quite slow [12], leading to significant numbers of message communications with the individual agents.

In this paper, we exploit the problem structure to obtain decentralized optimization problems. In such a scheme, the ISO transmits a price signal to the individual agents, who in turn solve their individual optimization problems, the solutions of which are communicated back to the ISO so they may update the price. However, in contrast to previous approaches, we impose that the agents also return the sensitivity of their solution to changes in the price. We show this can be computed analytically for the GenCos and DSOs. With this information, we propose that the ISO solves its market clearing problem by solving an implicit complementarity problem (ICP) as introduced in Curtis and Raghunathan [13]. In this work, a semismooth equation approach is proposed for accelerating a dual decomposition algorithm for solving structured quadratic programs. We demonstrate through numerical experiments that

our approach leads to orders of magnitude fewer function evaluations as compared to a subgradient method.

C. Organization of the Paper

Models of the market agents and the notions of competitive equilibrum are presented in §II. An implicit complementarity formulation of the ISO's market-clearing problem is presented in §III. A semismooth formulation and algorithm are described in §IV. Numerical results demonstrating the efficacy of the method are presented in §V followed by conclusions in §VI.

II. Competitive Equilibrium

In this section, we describe the optimization problems related to each of the market agents: generation companies (GenCos), Distribution System Operators (DSOs), and the Independent System Operator (ISO). Based on these, we present the notion of competitive equilibrium and social welfare maximization. In what follows, $\mathcal{N}$ denotes the set of buses in the transmission network of the ISO while $\mathcal{N}^G$ and $\mathcal{N}^D$ (with $\mathcal{N} = \mathcal{N}^G \cup \mathcal{N}^D$) respectively denote the nodes connected to GenCos and DSOs. Further, $\mathcal{L}$ denotes the set of lines in the transmission network.

A. Generation Company (GenCo)

The generation company located at node $i \in \mathcal{N}^G$ chooses its optimal power generation level $P_i^*(\lambda_i)$ given the nodal price λ_i from the ISO by solving the optimization problem

$$P_i^*(\lambda_i) = \arg\min_P \; c_i(P) - \lambda_i P \tag{1a}$$

$$\text{s.t. } \underline{P}_i^G \leq P \leq \overline{P}_i^G, \tag{1b}$$

where $\underline{P}_i^G$ and $\overline{P}_i^G$ are the mimimum and maximum generation levels. We assume the following on the cost function of the GenCo, which implies that (1) has a unique solution.

Assumption 1. *The function c_i is strictly convex.*

B. Distribution System Operator (DSO)

The DSO located at node $i \in \mathcal{N}^D$ chooses its optimal power consumption level $-P_i^*(\lambda_i)$ given the nodal price λ_i from the ISO by solving the optimization problem

$$P_i^*(\lambda_i) = \arg\min_P \; -\lambda_i P - u_i(-P) \tag{2a}$$

$$\text{s.t. } \underline{P}_i^D \leq -P \leq \overline{P}_i^D, \tag{2b}$$

where u_i is the utility function of the DSO and $\underline{P}_i^D$ and $\overline{P}_i^D$ are minimum and maximum power consumption levels. Note that $P_i^*(\lambda_i)$ is negative since it represents power consumption as opposed to power generation. We assume the following on the utility function of the DSO which ensures that (2) has a unique solution.

Assumption 2. *The function u_i is strictly concave.*

C. Independent System Operator (ISO)

The ISO is responsible for maintaining balance between the GenCos and DSOs, and ensuring that the power flows in the network are within certain limits. Given a vector of nodal prices $\boldsymbol{\lambda} \in \mathbb{R}^{|\mathcal{N}|}$, the ISO chooses the optimum power injection levels at the nodes by solving the optimization problem,

$$\boldsymbol{P}^{\text{ISO}}(\boldsymbol{\lambda}) = \arg\min_{\boldsymbol{P}} \; \boldsymbol{\lambda}^T \boldsymbol{P} \tag{3a}$$

$$\text{s.t. } \mathbf{1}^T \boldsymbol{P} = 0 \tag{3b}$$

$$-\overline{\boldsymbol{P}} \leq \boldsymbol{A}\boldsymbol{P} \leq \overline{\boldsymbol{P}} \tag{3c}$$

where $\overline{\boldsymbol{P}} \in \mathbb{R}^{|\mathcal{L}|}$ denotes the vector of power limits on the lines in the network, $\mathbf{1} \in \mathbb{R}^{|\mathcal{N}|}$ is a vector of all ones, and $\boldsymbol{A}$ is the matrix of power distribution factors for the ISO's transmission network. The constraint (3b) imposes power balance between the GenCos and DSOs. The DC power flow model appears in (3c) through the power distribution factors [14].

D. Competitive Equilibrium

A pair $(\widehat{\boldsymbol{P}}, \widehat{\boldsymbol{\lambda}})$ is said to achieve *competitive (or Walrasian)* equilibrium for an electricity market if:

(a) $\widehat{P}_i = P_i^*(\widehat{\lambda}_i) \; \forall \; i \in \mathcal{N}^G$,
(b) $\widehat{P}_i = P_i^*(\widehat{\lambda}_i) \; \forall \; i \in \mathcal{N}^D$, and
(c) $\widehat{\boldsymbol{P}} = \boldsymbol{P}^{\text{ISO}}(\widehat{\boldsymbol{\lambda}})$.

By the well-known first and second fundamental theorems of welfare economics [15], we have the following.

- A competitive equilibrium is Pareto optimal.
- Every Pareto optimal allocation can be decentralized as a competitive equilibrium.

By the second fundamental theorem of welfare economics [8], [15], a competitive equilibrium can be characterized by maximizing social welfare given as

$$\min_{\boldsymbol{P}} \sum_{i \in \mathcal{N}^G} c_i(P_i) - \sum_{i \in \mathcal{N}^D} u_i(-P_i) \tag{4a}$$

$$\text{s.t. } \mathbf{1}^T \boldsymbol{P} = 0 \tag{4b}$$

$$-\overline{\boldsymbol{P}} \leq \boldsymbol{A}\boldsymbol{P} \leq \overline{\boldsymbol{P}} \tag{4c}$$

$$\underline{P}_i^G \leq P_i \leq \overline{P}_i^G \; \forall \, i \in \mathcal{N}^G \tag{4d}$$

$$\underline{P}_i^D \leq -P_i \leq \overline{P}_i^D \; \forall \, i \in \mathcal{N}^D. \tag{4e}$$

III. Decentralized Market Formulation

We develop a decentralized market formulation based on the ISO's optimization problem (3). For ease of presentation, we represent the power balance constraint in (3b) as the pair of inequalities

$$\mathbf{1}^T \boldsymbol{P} \geq 0 \text{ and } -\mathbf{1}^T \boldsymbol{P} \geq 0. \tag{3b'}$$

Introducing multipliers $\underline{\xi}$ and $\overline{\xi}$ for the power balance constraints in (3b') and $\underline{\boldsymbol{\zeta}}$ and $\overline{\boldsymbol{\zeta}} \in \mathbb{R}^{|\mathcal{L}|}$ for the line limit

constraints in (3c), the optimality conditions [16] for the ISO's problem are

$$\boldsymbol{\lambda} = (\underline{\xi} - \overline{\xi})\mathbf{1} + \boldsymbol{A}^T(\underline{\zeta} - \overline{\zeta}) \tag{5a}$$
$$0 \le \underline{\xi} \perp (\mathbf{1}^T\boldsymbol{P}) \ge 0 \tag{5b}$$
$$0 \le \overline{\xi} \perp (-\mathbf{1}^T\boldsymbol{P}) \ge 0 \tag{5c}$$
$$0 \le \underline{\zeta} \perp (\boldsymbol{A}\boldsymbol{P} + \overline{\boldsymbol{P}}) \ge 0 \tag{5d}$$
$$0 \le \overline{\zeta} \perp (-\boldsymbol{A}\boldsymbol{P} + \overline{\boldsymbol{P}}) \ge 0 \tag{5e}$$

where for a pair of vectors $\{a, b\}$ the expression $0 \le a \perp b \ge 0$ represents the conditions $a_i \ge 0$, $b_i \ge 0$, and $a_i b_i = 0$ for all i. The constraints in (5b)–(5e) are the so-called *complementarity constraints* [16]. At a competitive equilibrium the conditions in (5) must hold with $\boldsymbol{P} = \boldsymbol{P}^*(\boldsymbol{\lambda})$. In other words, the ISO's market-clearing problem can be posed as the *implicit complementarity problem* (ICP)

$$0 \le \boldsymbol{\nu} \perp F(\boldsymbol{\nu}) \ge 0, \tag{6}$$

$$\text{where } \boldsymbol{\nu} = \begin{pmatrix} \underline{\xi} \\ \overline{\xi} \\ \underline{\zeta} \\ \overline{\zeta} \end{pmatrix} \text{ and } F(\boldsymbol{\nu}) = \begin{pmatrix} \mathbf{1}^T\boldsymbol{P}^*(\boldsymbol{\lambda}) \\ -\mathbf{1}^T\boldsymbol{P}^*(\boldsymbol{\lambda}) \\ \boldsymbol{A}\boldsymbol{P}^*(\boldsymbol{\lambda}) + \overline{\boldsymbol{P}} \\ -\boldsymbol{A}\boldsymbol{P}^*(\boldsymbol{\lambda}) + \overline{\boldsymbol{P}} \end{pmatrix} \tag{7}$$

while $\boldsymbol{\lambda}$ satisfies (5a). We call this an *implicit* complementarity problem since $\boldsymbol{P}^*(\boldsymbol{\lambda})$ is obtained by solving a set of optimization problems. Observe that the evaluation of $\boldsymbol{P}^*(\boldsymbol{\lambda})$ only requires communication with the GenCos and DSOs through communication of the price vector $\boldsymbol{\lambda}$. Thus, the ICP (6) has the desired property of decoupling by agents and allows the agents to mainatain *privacy* of their optimization problem.

The following theorem formalizes the equivalence between the ICP (6) and the competitive equilibrium.

Theorem 1. *The following are equivalent:*
(a) $(\widehat{\boldsymbol{P}}, \widehat{\boldsymbol{\lambda}})$ *is a competitive equilibrium;*
(b) $\widehat{\boldsymbol{\nu}}$ *solves* (6) *with* $\widehat{\boldsymbol{\lambda}} = (\widehat{\underline{\xi}} - \widehat{\overline{\xi}})\mathbf{1} + \boldsymbol{A}^T(\widehat{\underline{\zeta}} - \widehat{\overline{\zeta}})$.

Proof. First, we show that (a) implies (b). Suppose (a) holds. From the definition of competitive equilibrium in §II-D, $\widehat{\boldsymbol{P}} = \boldsymbol{P}^*(\widehat{\boldsymbol{\lambda}})$. Since $\widehat{\boldsymbol{P}}$ solves the ISO's problem (3), we have that there exists multipliers $(\widehat{\underline{\xi}}, \widehat{\overline{\xi}}, \widehat{\underline{\zeta}}, \widehat{\overline{\zeta}})$ satisfying the optimality conditions in (5) with $\boldsymbol{P} = \boldsymbol{P}^*(\widehat{\boldsymbol{\lambda}})$. Thus, (b) holds. Now, suppose (b) holds. By the preceding arguments we have that first order stationarity conditions of the ISO's problem (3) holds. Since (3) is convex, a first order stationary point is also a minimizer [16]. This completes the proof. □

IV. Semismooth Equation Approach

We describe the semismooth equation approach of Curtis and Raghunathan [13] for solving the ICP. We rewrite the ICP (6) using the Fischer-Burmeister function [17]

$$\Phi^{\mathrm{FB}}(\boldsymbol{\nu}) = \begin{pmatrix} \phi(\nu_1, F_1(\boldsymbol{\nu})) \\ \vdots \\ \phi(\nu_m, F_m(\boldsymbol{\nu})) \end{pmatrix}, \tag{8}$$

where, given scalars a and b, function ϕ has the form

$$\phi(a, b) = \sqrt{a^2 + b^2} - a - b. \tag{9}$$

Clearly, this latter function satisfies

$$\phi(a, b) = 0 \iff \{a \ge 0,\ b \ge 0,\ \text{and } ab = 0\}. \tag{10}$$

At each iteration k of the semismooth Newton algorithm [18] the step $d\boldsymbol{\nu}^k$ is obtained as the solution of,

$$\Phi^{\mathrm{FB}}(\boldsymbol{\nu}^k) + H^k d\boldsymbol{\nu}^k = 0, \tag{11}$$

where H^k represents the first-order variation of the function Φ^{FB} at the point $\boldsymbol{\nu}^k$. We postpone the discussion on the computation of the matrix H^k to §IV-D and instead focus on the local convergence property and algorithmic details. The step $d\boldsymbol{\nu}^k$ obtained by solving (11) is called the *Semismooth Newton* step.

A. Fast Local Convergence

Semismooth functions such as Φ^{FB} are almost everywhere differentiable except on a set of measure zero [18]. Further, at points of non-differentiability, Φ^{FB} is directionally differentiable and can be approached through a sequence of differentiable points. Consequently, for any sequence of directions $d\boldsymbol{\nu} \to 0$ with associated Jacobians $H \in \partial\Phi(\boldsymbol{\nu} + d\boldsymbol{\nu})$ and directional derivatives $(\Phi^{\mathrm{FB}})'(\boldsymbol{\nu}; d\boldsymbol{\nu})$, we have that

$$\|H d\boldsymbol{\nu} - (\Phi^{\mathrm{FB}})'(\boldsymbol{\nu}; d\boldsymbol{\nu})\| = o(\|d\boldsymbol{\nu}\|). \tag{12}$$

This Taylor-series-like property is sufficient to show that iterations defined by (11) can converge locally superlinearly.

Theorem 2 ([18])**.** *Suppose that F is continuously differentiable and $\boldsymbol{\nu}^*$ satisfies $\Phi^{\mathrm{FB}}(\boldsymbol{\nu}^*) = 0$ such that all $H \in \partial\Phi^{\mathrm{FB}}(\boldsymbol{\nu}^*)$ are non-singular. Then, for any $\boldsymbol{\nu}^k$ in a sufficiently small neighborhood of $\boldsymbol{\nu}^*$, it follows that $\|\boldsymbol{\nu}^{k+1} - \boldsymbol{\nu}^*\| \le C\|\boldsymbol{\nu}^k - \boldsymbol{\nu}^*\|^{1+\gamma}$ for some $C > 0$ and $\gamma > 0$.*

In the present setting, F is not continuously differentiably, only piecewise differentiable (PC^1) since $P_i^*(\cdot)$ are PC^1 [13]. The main result in [13] proves local superlinear convergence for $F \in PC^1$. Hence, the semismooth newton iteration [13] converges fast locally, unlike a conventional subgradient approach. We provide numerical evidence for this in §V.

B. Algorithm

To promote global convergence, we employ a line-search based on the merit function $\Psi^{\mathrm{FB}}(\boldsymbol{\nu}) := \|\Phi^{\mathrm{FB}}(\boldsymbol{\nu})\|^2$. Observe that the minimum of $\Psi^{\mathrm{FB}}(\boldsymbol{\nu})$ is 0 corresponding to a solution of the ICP (6). Thus, reduction of the merit function $\Psi^{\mathrm{FB}}(\boldsymbol{\nu})$ can be used to certify that the steps of the algorithm ultimately decrease the distance to a solution of the ICP. Given a direction $d\boldsymbol{\nu}^k$, the step length α^k is determined as the largest $\alpha^k \in (0, 1]$ such that the sufficient decrease condition

$$\Psi^{\mathrm{FB}}(\boldsymbol{\nu}^k + \alpha^k d\boldsymbol{\nu}^k) \le \Psi^{\mathrm{FB}}(\boldsymbol{\nu}^k) + \eta\alpha^k \nabla\Phi^{\mathrm{FB}}(\boldsymbol{\nu}^k)^T d\boldsymbol{\nu}^k \tag{13}$$

holds where $\eta \in (0, 1)$; e.g., one typically chooses $\eta = 10^{-4}$. The step-length α^k may be obtained using a backtracking line-search starting from value of 1 and multiplying by a

constant factor $\rho \in (0,1)$ until the suffcient decrease condition holds (13). The complete steps of the algorithm are provided in Algorithm 1. Steps 5 and 6 require the computation of the sensitivity of the optimal solution to the variation in the price signal $\frac{\partial P_i^*(\lambda_i)}{\partial \lambda_i}$ in addition to the optimal solution. We provide the analytical formula for this in §IV-C. The computation of the matrix H^k in Step 7 is described in §IV-D.

Algorithm 1: Semismooth Newton Algorithm (SSN)

1 Choose a convergence tolerance $\epsilon \in (0,1)$ and an initial guess $\boldsymbol{\nu}^0 = (\underline{\xi}^0, \overline{\xi}^0, \underline{\zeta}^0, \overline{\zeta}^0)$. Choose $\{\eta, \rho\} \subset (0,1)$.
2 Set $k = 0$.
3 **repeat**
4 Set $\boldsymbol{\lambda}^k$ according to (5a).
5 For $i \in \mathcal{N}^G$, set $(P_i^*(\lambda_i^k), \frac{\partial P_i^*}{\partial \lambda_i}(\lambda_i^k))$ from {(1),(14)}.
6 For $i \in \mathcal{N}^D$, set $(P_i^*(\lambda_i^k), \frac{\partial P_i^*}{\partial \lambda_i}(\lambda_i^k))$ from {(2),(15)}.
7 Compute H^k using (16) and $d\boldsymbol{\nu}^k$ using (11).
8 Find the smallest integer $n \geq 0$ such that (13) holds for $\alpha^k = \rho^n$.
9 Set $\boldsymbol{\nu}^{k+1} = \boldsymbol{\nu}^k + \alpha^k d\boldsymbol{\nu}^k$ and $k = k+1$
10 **until** $\|\phi^{\mathrm{FB}}(\boldsymbol{\nu}^k)\|_\infty \leq \epsilon$

C. Computing $\frac{\partial P_i^*}{\partial \lambda_i}(\lambda_i)$

For the optimization problems (1) and (2) the price signal λ_i is a parameter that occurs in the objective function. The theory of parametric sensitivity analysis of nonlinear programs [19] can be applied to obtain the sensitivity of the optimal solution as described below. (For simplicity in what follows, we assume strict complementarity of the GenCo and DSO problems. For more general situations, see [13].) For the GenCos ($i \in \mathcal{N}^G$),

$$\frac{\partial P_i^*}{\lambda_i}(\lambda_i^k) = \begin{cases} \frac{1}{c_i''(P_i^*(\lambda_i^k))} & \text{if } \underline{P}_i^G < P_i^*(\lambda_i^k) < \overline{P}_i^G \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $c_i''(P_i^*(\lambda_i^k))$ denotes the second derivative of the cost function at the power generation level $P_i^*(\lambda_i^k)$. For the DSOs ($i \in \mathcal{N}^D$),

$$\frac{\partial P_i^*}{\lambda_i}(\lambda_i^k) = \begin{cases} \frac{-1}{u_i''(-P_i^*(\lambda_i^k))} & \text{if } \underline{P}_i^D < -P_i^*(\lambda_i^k) < \overline{P}_i^D \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where $u_i''(-P_i^*(\lambda_i^k))$ denotes the second derivative of the utility function at the power consumption level $P_i^*(\lambda_i^k)$. From (14) and (15) we have that there exists sensitivity to price signal only when the optimal solutions to (1) and (2) lie within their respective bounds. Consequently, the function $\boldsymbol{P}^*$ is not smooth; in fact, it is known to be *semismooth* [13].

D. Computing H^k

The matrix H^k is defined as

$$\begin{aligned} H^k &= D_{\boldsymbol{\nu}}^k + D_F^k \nabla F(\boldsymbol{\nu}^k)^T \\ &= D_{\boldsymbol{\nu}}^k + D_F^k \boldsymbol{B}^k \begin{bmatrix} -\mathbf{1} & \mathbf{1} & -\boldsymbol{A}^T & \boldsymbol{A}^T \end{bmatrix} \end{aligned} \tag{16}$$

where $D_{\boldsymbol{\nu}}^k$ and D_F^k are diagonal matrices. Introducing the set $\beta^k = \{j \mid \nu_j^k = 0 = F_j(\boldsymbol{\nu}^k)\}$, they can be obtained as

$$[D_{\boldsymbol{\nu}}^k]_{jj} = \begin{cases} \left(\frac{\nu_j^k}{\|(\nu_j^k, F_j^k)\|} - 1 \right) & \forall\, j \notin \beta^k \\ \left(\frac{z_j}{\|(z_j, z^T \nabla F_j^k)\|} - 1 \right) & \forall\, j \in \beta^k \end{cases}$$

$$[D_F^k]_{jj} = \begin{cases} \left(\frac{F_j^k}{\|(\nu_j^k, F_j^k)\|} - 1 \right) & \forall\, j \notin \beta^k \\ \left(\frac{z^T \nabla F_j^k}{\|(z_j, z^T \nabla F_j^k)\|} - 1 \right) & \forall\, j \in \beta^k \end{cases}$$

where ∇F_j^k denotes the gradient of the $F_j(\cdot)$ evaluated at $\boldsymbol{\nu}^k$ and z is chosen such that $z_j = 1$ for $j \in \beta^k$ and 0 otherwise [20]. The matrix

$$\boldsymbol{B}^k = \operatorname{diag}\left(\frac{\partial P_1^*}{\partial \lambda_1}(\lambda_1^k), \ldots, \frac{\partial P_{|\mathcal{N}|}^*}{\partial \lambda_{|\mathcal{N}|}}(\lambda_{|\mathcal{N}|}^k) \right)$$

is a diagonal matrix of the sensitivities of the optimum response of the agents to their respective price signal λ_i.

V. Numerical Results

In this section, we demonstrate the numerical efficiency of the semismooth Newton approach in Algorithm 1. We compare the algorithm to a standard subgradient algorithm [12] for computing the competitive equilibrium. The iteration of the subgradient algorithm is

$$\boldsymbol{\nu}^{k+1} = \max\left\{0, \boldsymbol{\nu}^k - \frac{1}{k+1} F(\boldsymbol{\nu}^k)\right\}. \tag{17}$$

This replaces Steps 7-9 in Algorithm 1. The update step in (17) only requires the optimal solution of the GenCos and DSOs, i.e., the computation of sensitivity information in Steps 5 and 6 of Algorithm 1 is avoided. The iterations are quite simple to implement and fit the decentralized framework very well. The typical number of iterations required to obtain a solution of tolerance ϵ is $O(\frac{1}{\epsilon^2})$ [12]. Thus, a large number of communication rounds are required between the ISO and the GenCos and DSOs to achieve convergence.

In the following, we consider 6 different sizes of transmission networks corresponding to IEEE 9-bus, 14-bus, 30-bus, 39-bus, 57-bus and 118-bus networks. The cost function for the GenCos is chosen as a strictly convex quadratic function, $c_i(P) = c_{1i}P + c_{2i}P^2$ where $c_{2i} > 0$. The values for the coefficients c_{1i} and c_{2i} are obtained from the data files. The utility function for the DSOs is chosen as a strictly concave quadratic function, $u_i(-P) = u_{i1}(-P) + u_{2i}P^2$ where $u_{2i} < 0$. The coefficient values u_{1i} and u_{2i} are generated randomly. The lower and upper bounds for demands of the DSOs are respectively taken to be 0.8 and 1.2 times the demand provided in the data file. Both algorithms were implemented in MATLAB and run on a Windows desktop with 3 GB RAM and a 3.0 GHz Intel Core 2 Quad processor.

	Subgradient		Semismooth		
Case #	# Iters	Time (s)	# Iters	# Fcn.	Time (s)
9-bus	100000	2.51	6	19	0.01
14-bus	100000	2.67	6	65	0.06
30-bus	100000	1.42	5	16	0.05
39-bus	43250	1.42	10	83	0.15
57-bus	100000	8.67	7	23	0.15
118-bus	100000	29.5	6	31	1.50

TABLE I
PERFORMANCE STATISTICS FOR THE ALGORITHMS ON DIFFERENT CASES. # ITERS - NUMBER OF ITERATIONS, # FCN. - NUMBER OF FUNCTION EVALUATIONS.

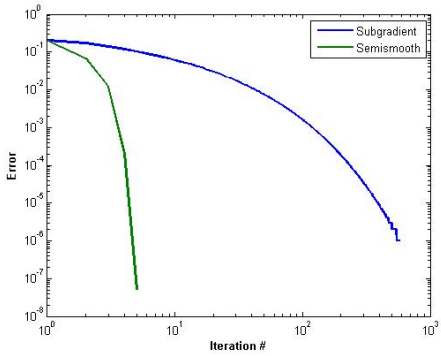

Fig. 1. Plot of error againt iteration index for the algorithms

Table I summarizes the performance statistics of the algorithms. The reported numbers are the average of 10 different runs. In each run, the cost functions, utility functions, and demands were generated randomly. Both algorithms were set to run until either the tolerance $\epsilon = 10^{-6}$ was reached or the limit on the number of iterations, namely 100000, was reached. From Table I, it is clear that the subgradient algorithm hits the maximum iteration limit on most problems. The proposed semismooth Newton Algorithm requires $\leq$10 iterations. The number of function evaluations required by semismooth Newton depends on the step size that is acceptable for the merit function decrease condition (13). As a consequence, the number of function evaluations is higher than the iteration count. Still, the number of function evaluations is 4 orders of magnitude smaller than for the subgradient method. The update for the semismooth Newton Algorithm requires solving a linear system (11) and is more expensive than that for the subgradient algorithm (17). Nevertheless, the gain in reduction of iterations using a Newton algorithm outweighs the increase in computational time per iteration. Figure 1 plots the typical progress of the error in satisfying ICP against the iteration index. Further, the convergence rate is indeed superlinear as predicted by Theorem 2 and is key to explaining the observed acceleration in convergence over the subgradient method.

VI. CONCLUSIONS & EXTENSIONS

In this paper, we have presented a novel semismooth Newton Algorithm for the computation of competitive equilibrium in electricity markets. The approach requires the GenCos and DSOs to communicate not only their optimal response to the price signal from the ISO but also the sensitivity of the optimal solution to the price signal. We also present analytical expressions for computing these values. The proposed approach is shown to be robust in converging to a tight tolerance of 10^{-6} and requires about 4 orders of magnitude fewer function evaluations than the subgradient algorithm.

The current paper makes two assumptions on DSOs: lumped model and no distributed generation. We will extend the proposed approach to DSOs where the electrical network of the DSO is also modeled and distributed generation is included. We will also investigate the applicability of the approach when the DSO's power flow is modeled using AC power flow equations. In this context, we will also explore the convex SDP relaxation [10] which has shown to have zero duality gap in a number of instances.

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